Econometrics of Commodity and Asset Pricing

Lecture 3

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ENSAE, Finance & Gestion des Risques, MiE, 2025 - 2026

Agenda - Lecture 3

I. Gaussian Affine Term Structure Models (GATSMs).

II. A New Perspective on GATSMs - Part I.

Chapter I

GAUSSIAN AFFINE TERM STRUCTURE MODELS

– Agenda –

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I.1 – INTRODUCTION

- **Risk-free** = with no default (and/or liquidity) risks.
- An important share of the term-structure literature pertains to the modelling of risk-free yields. Some models involve macro factors (in w_t). Some do not ("yield-only" models).
- Basic formula (see Lecture 1):

$$B(t,h) = \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}} \exp(-r_{t+1} - \cdots - r_{t+h-1}) \quad \text{(I.1)}$$

$$R(t,h) = -\frac{1}{h}\log B(t,h). \quad \text{(I.2)}$$

- Models used to forecast yields, or to compute **term premiums (TP)** = components of yields that would not exist if investors were not risk-averse. If this was the case (**Expectation Hypothesis, EH**), then $M_{t,t+1} = \exp(-r_t) \Rightarrow$ "$\mathbb{P} \equiv \mathbb{Q}$" and $B(t,h)$ would become:

$$\exp(-r_t)\mathbb{E}_t \exp(-r_{t+1} - \cdots - r_{t+h-1}). \quad \text{(I.3)}$$

implying that the maturity-h yield-to-maturity would be:

$$R^{EH}(t,h) = -\frac{1}{h}\log\left(\mathbb{E}_t \exp(-r_t - \cdots - r_{t+h-1})\right) \approx \frac{1}{h}\mathbb{E}_t(r_t + \cdots + r_{t+h-1}).$$

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$\Rightarrow$ Term premium given by:

$$TP_{t,h} = \underbrace{-\frac{1}{h}\log \mathbb{E}_t^{\mathbb{Q}} \exp(-r_{t+1} - \cdots - r_{t+h-1})}_{=R(t,h)} - \underbrace{-\frac{1}{h}\log \mathbb{E}_t \exp(-r_{t+1} - \cdots - r_{t+h-1})}_{=R^{EH}(t,h)}. \tag{I.4}$$

- **Interpretation of $TP_{t,h}$:**
  - Under EH, investors are willing to buy a maturity-h bond as long as its expected return is – up to Jensen's inequality – equal to the average of future short-term rates (Hence the definition of $R^{EH}(t,h)$).
  - $TP_{t,h} > 0$ means that, due to risk aversion, investors are willing to buy the maturity-h bond only if its return is, on average, higher than expected future short-term rates.
  - Reflects the fact that investors consider that long-term bonds tend to lose value in "bad states of the world" (see Remark I.1).

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Remark I.1 (Risk premiums)

- *We know that, under the AAO, the following relation is satisfied by any asset j:*

$$p_{jt} = \mathbb{E}_t(M_{t,t+1} p_{j,t+1}).$$

- *It can be rewritten:*

$$p_{jt} = \mathbb{C}ov_t(M_{t,t+1}, p_{j,t+1}) + \mathbb{E}_t(M_{t,t+1})\mathbb{E}_t(p_{j,t+1})$$

or

$$p_{jt} = \underbrace{\exp(-r_t)\mathbb{E}_t(p_{j,t+1})}_{=p_{jt}^{EH}} + \underbrace{\mathbb{C}ov_t(M_{t,t+1}, p_{j,t+1})}_{\textit{Risk premium}}. \tag{I.5}$$

- *If investors were risk-neutral, then we would have $p_{jt} = p_{jt}^{EH}$.*
- *For risk-averse investors, the SDF is high (resp. low) in "bad (resp. good) states of the world", namely when the payoff $p_{j,t+1}$ (its future value) is low (resp. high).*

$\Rightarrow$ *$p_{jt} < p_{jt}^{EH}$ if asset j tends to pay less in bad states of the world (i.e. if $\mathbb{C}ov_t(M_{t,t+1}, p_{j,t+1}) < 0$).*

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I.2 – AFFINE TERM STRUCTURE MODELS

- What do we get when w_t is affine (in general, under $\mathbb{P}$ and $\mathbb{Q}$)? If the nominal short-term rate $r_t = \omega_0 + \omega_1' w_t$, then:

$$\begin{aligned} B(t,h) &= \mathbb{E}_t^{\mathbb{Q}} \exp(-r_t - \cdots - r_{t+h-1}) \\ &= \exp(-h\omega_0 - \omega_1' w_t)\mathbb{E}_t^{\mathbb{Q}} \exp(-\omega_1' w_{t+1} - \cdots - \omega_1' w_{t+h-1}). \end{aligned}$$

The (blue) expectation is easily computed using the recursive equations of a multi-horizon conditional Laplace transform (with a particular reverse ordering), leading to:

$$R(t,h) = -\frac{1}{h} \log B(t,h) = C_h' w_t + D_h. \tag{I.6}$$

- It is easily seen that we can also get:

$$R^{EH}(t,h) = C_h^{EH\prime} w_t + D_h^{EH}. \tag{I.7}$$

- Moreover, if inflation $\pi_t = \bar{\omega}_0 + \bar{\omega}_1' w_t$, then real yields are given by:

$$\bar{B}(t,h) = \mathbb{E}_t^{\mathbb{Q}} \exp(-r_t - \cdots - r_{t+h-1} + \pi_{t+1} + \cdots + \pi_{t+h})$$

which also leads to:

$$\bar{R}(t,h) = -\frac{1}{h} \log \bar{B}(t,h) = \bar{C}_h' w_t + \bar{D}_h. \tag{I.8}$$

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- Eqs. (I.6) and (I.7) imply that term premiums are affine in w_t (see Eq. I.4). Specifically:

$$TP(t,h) = R(t,h) - E^{EH}(t,h) = D_h - D_h^{EH} + (C_h - C_h^{EH})' w_t.$$

- Expected excess returns resulting from holding zero-coupon bonds are also affine in w_t.

  a) Indeed, holding a maturity-h zero-coupon bond for one period provides the following expected gross return:

$$\mathbb{E}_t\left(\frac{B(t+1,h-1)}{B(t,h)}\right) = \mathbb{E}_t\left(\exp(D_{h-1} - D_h + C'_{h-1} w_{t+1} - C'_h w_t)\right),$$

  which is clearly exponential affine in w_t if w_t is an affine process.

  b) Therefore, the expected excess return, that is:

$$\log \mathbb{E}_t\left(\frac{B(t+1,h-1)}{B(t,h)}\right) - r_t$$

  is also affine in w_t in this context.

- The fact that excess returns are affine in this context is exploited in the estimation approach proposed by Adrian, Crump and Moench (2013).

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- Moreover, conditional expectations of future interest rates (real or nominal) and of term premiums are affine in w_t. In particular:

$$\mathbb{E}_t[R(t+k,h)] = \mathbb{E}_t[C_h{}' w_{t+k} + D_h] = C_h{}' \mathbb{E}_t(w_{t+k}) + D_h, \qquad (I.9)$$

and $\mathbb{E}_t(w_{t+k})$ is affine in w_t (see Lecture 1). This can notably be used at the estimation stage, if one wants to fit survey data.

- Conditional variances of future interest rates (real or nominal) and of term premiums are affine in w_t. In particular:

$$\mathbb{V}ar_t[R(t+k,h)] = \mathbb{V}ar_t[C_h{}' w_{t+k} + D_h] = C_h{}' \mathbb{V}ar_t(w_{t+k}) C_h, \qquad (I.10)$$

where the components of $\mathbb{V}ar_t(w_{t+k})$ (and therefore $\mathbb{V}ar_t[R(t+k,h)]$) is affine in w_t (see Lecture 1). This can notably be used at the estimation stage, if one wants to fit (proxies of) conditional variances.

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- In an affine model, the maximum Sharpe ratio is easily computed. Eq. (I.5) implies that:

$$\mathbb{E}_t \underbrace{\left(\frac{p_{j,t+1}}{p_{j,t}} - \exp(r_t) \right)}_{=xs_{j,t+1}, \text{excess return}} = -\exp(r_t) \mathbb{C}ov_t \left(M_{t,t+1}, \frac{p_{j,t+1}}{p_{j,t}} \right),$$

and, using $|\mathbb{C}ov(X,Y)| \leq \sqrt{\mathbb{V}ar(X)\mathbb{V}ar(Y)}$, we get the Hansen-Jagannathan (1991) bound:

$$\underbrace{\frac{\mathbb{E}_t(xs_{j,t+1})}{\sqrt{\mathbb{V}ar_t(xs_{j,t+1})}}}_{\text{Sharpe ratio}} \leq \underbrace{\frac{\sqrt{\mathbb{V}ar_t(M_{t,t+1})}}{\mathbb{E}_t(M_{t,t+1})}}_{\text{Maximum Sharpe ratio}} . \tag{I.11}$$

- If the SDF is given by $M_{t,t+1} = \exp[-r_t + \alpha_t' w_{t+1} - \psi_t(\alpha_t)]$ (see Lecture 1), and using that:

$$\mathbb{E}_t(M_{t,t+1}^2) = \exp(-2\, r_t + \psi_t(2\alpha_t) - 2\, \psi_t(\alpha_t))$$

we get:

$$\text{Maximum Sharpe ratio} = \sqrt{\exp\left[\psi_t(2\alpha_t) - 2\, \psi_t(\alpha_t)\right] - 1}.$$

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I.3 – UNIVARIATE GAUSSIAN ATSMs

Historical dynamics:

- We consider an economy, in a **dynamic discrete-time setting**, between dates 0 and T. The new information in the economy at date t is denoted by w_t and the overall information at date t is $\underline{w}_t = (w_t, w_{t-1}, \ldots, w_0)$. It is the (common) **information judged relevant** by each investor to price assets.

- w_t is called a **factor or a state vector**, and it may be observable, partially observable or unobservable by the econometrician. The size of w_t is K.

- $w_t =$ **observable** $\rightarrow$ interest rates of different maturities, inflation rate, gross domestic product, ...

- $w_t =$ **non observable** $\rightarrow$ level, slope and curvature factors, market regimes (using regime-switching models), stochastic volatility, jumps (market crashes), ...

- $w_t =$ **partially observable** $\rightarrow$ $w_t = (w_{1,t}, w_{2,t})'$ where $w_{1,t}$ is observable and $w_{2,t}$ is not.

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Historical dynamics:

- The **historical dynamics** of w_t is defined by the joint distribution of $\underline{w}_T$, denoted by $\mathbb{P}$, or by the conditional probability density function (p.d.f.):

$$f_t(w_{t+1}|\underline{w}_t)\,,$$

or by the **conditional Laplace transform** (L.T.):

$$\varphi_t(u|\underline{w}_t) = \varphi_t(u) = \mathbb{E}[\exp(u' w_{t+1})|\underline{w}_t] = \mathbb{E}_t[\exp(u' w_{t+1})]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^K$ (containing zero). We also introduce the **conditional Log-Laplace transform**:

$$\psi_t(u|\underline{w}_t) = \psi_t(u) = \text{Log}[\varphi_t(u|\underline{w}_t)]\,.$$

- Let us assume that $K = 1$ and that the (non observable) factor w_{t+1} is a Gaussian AR(1) process of the following type:

$$w_{t+1} = \nu + \varphi\, w_t + \sigma\varepsilon_{t+1}\,, \tag{I.12}$$

where ε_{t+1} is a Gaussian white noise with $\mathcal{N}(0,1)$ distribution.

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Historical dynamics:

- $\mathbb{E}_t[w_{t+1}] = \nu + \varphi w_t$, $\mathbb{V}ar_t[w_{t+1}] = \sigma^2$, $w_{t+1} \mid w_t \sim N(\nu + \varphi w_t, \sigma^2)$
- $w^e_{t+k\,|\,t} := \mathbb{E}_t[w_{t+k}] = (1 + \varphi + \ldots + \varphi^{k-1})\,\nu + \varphi^k\, w_t$ (under $\mathbb{P}$).
- Under stationarity (i.e., $|\varphi| < 1$): $\mathbb{E}[w_t] = \dfrac{\nu}{1-\varphi}$, $\mathbb{V}ar[w_t] = \dfrac{\sigma^2}{1-\varphi^2}$,

$$\Rightarrow \quad w_t \sim N\left(\frac{\nu}{1-\varphi}, \frac{\sigma^2}{1-\varphi^2}\right), \text{ with } \lim_{k\to+\infty} \mathbb{E}_t[w_{t+k}] = \mathbb{E}[w_t].$$

- If $Y \sim N(\mu, \omega^2)$, then:

$$\varphi(u) = \mathbb{E}[\exp(uY)] = \exp\left(u\mu + \frac{1}{2}\,u^2\omega^2\right).$$

- This means that:

$$\varphi_t(u|\underline{w}_t) = \varphi_t(u) = \exp\left[u(\nu + \varphi w_t) + \frac{1}{2}\,u^2\,\sigma^2\right],$$

and

$$\mathbb{E}[\exp(uw_t)] = \exp\left[u\left(\frac{\nu}{1-\varphi}\right) + \frac{1}{2}\,u^2\,\frac{\sigma^2}{1-\varphi^2}\right].$$

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The Stochastic Discount Factor:

- We price assets (ZCBs in our case!) following the no-arbitrage principle.

- The development of the zero-coupon bond (no arbitrage) pricing model is characterized:
  - *i)* after the specification of the historical distribution (presented above),
  - *ii)* by the parametric specification of a positive stochastic discount factor (SDF) $M_{t,t+1}$, for the period $(t, t+1)$.

- The price p_t at t of a financial asset (basic asset, derivative, ...) paying p_T at T is:

$$p_t = \mathbb{E}\left[M_{t,t+1} \cdot \ldots \cdot M_{T-1,T}\, p_T \,|\, \underline{w}_t\right] = \mathbb{E}_t\left[M_{t,T}\, p_T\right].$$

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The Stochastic Discount Factor:

- We choose a SDF which is exponential-affine in the state variable w_{t+1}, that is (equivalently), in its noise ε_{t+1} :

$$M_{t,t+1} = \exp\left[-\beta - \alpha\, w_t + \Gamma_t\, \varepsilon_{t+1} - \frac{1}{2}\,\Gamma_t^2\right] \; : \tag{I.13}$$

  a) the coefficients α and β are path independent (constant!);
  b) $\Gamma_t = \Gamma(w_t) = (\gamma_o + \gamma\, w_t)$ is a stochastic risk correction coefficient, also called **Market Price of Factor Risk** [see following sections].

- Now, the absence of arbitrage restriction on the ZCB with unitary residual maturity requires:

$$\mathbb{E}_t(M_{t,t+1}) = B(t,1) = \exp(-r_t),$$

where r_t is the (predetermined) short-term interest rate between t and $t+1$. This condition implies the relation $r_t = \beta + \alpha w_t$, and this means that, under the absence of arbitrage opportunities, the SDF can be written as:

$$M_{t,t+1} = \exp\left[-r_t + \Gamma_t\, \varepsilon_{t+1} - \frac{1}{2}\,\Gamma_t^2\right] = \exp(-r_t)\, d_{t+1}^{\mathbb{Q}}(w_{t+1})\,.$$

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The Risk Premium:

- In order to give an interpretation of the risk-correction coefficient Γ_t, we consider the following definition of risk premium:

Definition I.1

If we denote by P_t the price at time t of a given asset, its risk premium between t and $t+1$ is:

$$\lambda_t = \log \mathbb{E}_t \left(\frac{P_{t+1}}{P_t} \right) - r_t = \log \mathbb{E}_t \exp(y_{t+1}) - r_t \,,$$

where $y_{t+1} = \log(P_{t+1}/P_t)$ denotes the one-period geometric return.

- We thus obtain an interpretation of Γ_t by means of the following:

Example I.1

If we consider an asset providing the payoff $P_{t+1} = \exp(-b\, w_{t+1})$ at $t+1$ $(b > 0)$, its price in t is given by:

$$\begin{aligned} P_t &= \mathbb{E}_t \left[M_{t,t+1}\, P_{t+1} \right] = \exp[-r_t - b(\nu + \varphi w_t) - b\sigma\Gamma_t + \frac{1}{2}(b\sigma)^2] \,, \\ \mathbb{E}_t P_{t+1} &= \exp[-b(\nu + \varphi w_t) + \frac{1}{2}(b\sigma)^2] \,, \quad \textit{and thus} \;\; \lambda_t = b\,\sigma\,\Gamma_t \,. \end{aligned}$$

For given $b > 0$ and $\sigma > 0$, we have that $\Gamma_t = 0$ implies $\lambda_t = 0$.

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The Affine Yield Curve Formula:

- With the specification of the SDF, we determine the price of a zero-coupon bond in the following way :

$$B(t,h) = \mathbb{E}_t \left[M_{t,t+1} \cdot \ldots \cdot M_{t+h-1,t+h} \right] ,$$

where $B(t,h)$ denotes the price at time t for a ZCB with residual maturity h.

Prop. I.1

Let us assume (w_t) follows under $\mathbb{P}$ the Gaussian AR(1) process (I.12) and that the one-period SDF is given by (I.13). Then the no-arbitrage price at date t of the zero-coupon bond with residual maturity h is:

$$B(t,h) = \exp(c_h w_t + d_h) , \quad h \geq 1 , \tag{I.14}$$

where c_h and d_h satisfies the recursive equations:

$$\begin{cases} c_h &= -\alpha + \varphi^* c_{h-1} , \\ d_h &= -\beta + c_{h-1}\nu^* + \frac{1}{2} c_{h-1}^2 \sigma^2 + d_{h-1} , \end{cases}$$

with $\varphi^* = (\varphi + \sigma\gamma)$, $\nu^* = (\nu + \gamma_o \sigma)$. The initial conditions of these recursive equations are:

- *at $h = 0$ we have $B(t,0) = 1$, implying the conditions $c_0 = 0$ and $d_0 = 0$.*
- *or, at $h = 1$ we have $B(t,1) = \exp(-r_t)$, implying the conditions $c_1 = -\alpha$ and $d_1 = -\beta$.*

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The Affine Yield Curve Formula:

- **Proof:** given that $M_{t,t+1}$ is exponential-affine in ε_{t+1} (i.e. w_{t+1}) and that the conditional Laplace transform of w_{t+1} is exponential-affine in the conditioning variable (w_t) we suggest that the ZCB pricing formula at date t be an exponential-affine function of w_t.
- We proceed in the following way: *a)* we suggest $B(t,h) = \exp(c_h\, w_t + d_h)$ and we (equivalently) rewrite the pricing formula in terms of the payoff $B(t+1, h-1)$:

$$\begin{aligned} B(t,h) &= \exp(c_h\, w_t + d_h) = \mathbb{E}_t[M_{t,t+1} \cdots M_{t+h-1,t+h}] \\ &= \mathbb{E}_t[M_{t,t+1} B(t+1, h-1)] \\ &= \mathbb{E}_t\left[\exp\left(-\beta - \alpha\, w_t + \Gamma_t\, \varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma_t^2\right) \exp(c_{h-1}\, w_{t+1} + d_{h-1})\right], \end{aligned}$$

- *b)* we do the algebra (calculating the conditional Laplace transform) obtaining:

$$\begin{aligned} & B(t,h) = \exp(c_h\, w_t + d_h) \\ = \;& \exp\left[-\beta - \alpha w_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1}\right] \times \mathbb{E}_t[\exp\left(\Gamma_t \varepsilon_{t+1} + c_{h-1} w_{t+1}\right)] \\ = \;& \exp\left[-\beta - \alpha w_t - \tfrac{1}{2}\Gamma_t^2 + d_{h-1} + c'_{h-1}(\varphi w_t + \nu)\right] \\ & \quad \times \mathbb{E}_t[\exp\left(\Gamma_t + \sigma c_{h-1})\varepsilon_{t+1}\right)] \\ = \;& \exp\Big[(-\alpha + \varphi c_{h-1} + c_{h-1}\sigma\gamma) w_t \\ & \quad + (-\beta + c_{h-1}\nu + \tfrac{1}{2}\, c_{h-1}^2\, \sigma^2 + \gamma_o c_{h-1}\sigma + d_{h-1})\Big], \end{aligned}$$

and by identifying the coefficients we find the recursive relation presented in Proposition I.1.

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The Affine Yield Curve Formula:

- The yields to maturity (continuously compounded spot rates) associated to the ZCB pricing formula are :

$$\begin{aligned} R(t,h) &= -\frac{1}{h}\log B(t,h) \\ &= -\frac{c_h}{h}w_t - \frac{d_h}{h}\,, \quad h \geq 1\,, \end{aligned}$$

  and they are **affine functions** of the factor w_t.
- For a given t and with h varying, $R(t,h)$ is the so-called affine term structure of interest rates, and the model is called Affine Term Structure Model (ATSM).
- (w_t) is a discrete-time Gaussian affine (the AR(1)) process: then we talk about Gaussian Discrete-Time ATSM.
- (w_t) is a scalar process : Univariate Gaussian ATSM.

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Excess Returns of Zero-Coupon Bonds:

Prop. I.2

Under the absence of arbitrage opportunity, and for a fixed maturity T, the one-period geometric zero-coupon bond return process:

$$\rho = [\,\rho(t,T), 0 \leq t \leq T\,], \text{ with}$$

$$\rho(t+1,T) = \log[B(t+1,\ T-t-1)] - \log[B(t,\ T-t)]\ ,$$

is given by:

$$\rho(t+1,\ T) = r_t - \tfrac{1}{2}\,\omega(t+1,\ T)^2 + \omega(t+1,\ T)\,\Gamma_t - \omega(t+1,\ T)\,\varepsilon_{t+1}\ ,$$

where $\omega(t+1,\ T) = -(\sigma c_{T-t-1})$.

- This means that the process ρ is such that:

$$\rho(t+1,T)\,|\,\underline{w_t} \ \sim\ N\left[\mu(t+1,\ T), \omega(t+1,\ T)^2\right]$$

where $\mu(t+1,\ T) = r_t - \frac{1}{2}\,\omega(t+1,\ T)^2 + \omega(t+1,\ T)\Gamma_t\ ,$

and $\omega(t+1,\ T)^2 = (\sigma c_{T-t-1})^2\ .$

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Excess Returns of Zero-Coupon Bonds:

- The associated risk premium between t and $t+1$, denoted by $\lambda_t(T)$, is:

$$\lambda_t(T) = \log \mathbb{E}_t \exp[\rho(t+1,\ T)] - r_t = \omega(t+1,\ T)\,\Gamma_t\,.$$

  and $\Gamma_t = (\gamma_o + \gamma w_t)$ plays (for any T) the role of a risk premium per unit of "risk" $\omega(t+1,\ T)$.

- In particular, for a fixed $\gamma \neq 0$, the variability of $\lambda_t(T)$ is driven by w_t.
- If we assume $\gamma = 0$ (i. e., $\Gamma_t = \gamma_o$), the risk correction coefficient and the risk premium of the zero-coupon bond become constants.
- Also note that, if $T = t+2$ and $w_t = r_t$, we have $\omega(t+1,\ T) = \sigma$ and we get the result of the Example I.1 for $b = 1$.
- We will see that this property of the excess bond return process gives the opportunity to easily estimate the model, and in particular (γ_o, γ).

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Risk Neutral Dynamics:

- Under the absence of arbitrage opportunity, there exist a probability $\mathbb{Q} \sim \mathbb{P}$ under which asset prices, evaluated with respect to some numeraire N_t, are martingales:

$$\frac{p_{j,t}}{N_t} = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{p_{j,t+1}}{N_{t+1}} \right].$$

- $\mathbb{Q}$ is be the probability (equivalent to $\mathbb{P}$) defined by the sequence of conditional densities:

$$d_{t+1}^{\mathbb{Q}} := \frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{N_{t+1}\, M_{t,t+1}}{N_t} > 0\,, \quad \mathbb{E}_t^{\mathbb{P}} \left[\frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} \right] = 1\,,$$
$$t \in \{0, \ldots, T-1\}\,.$$

- If we consider as numeraire the money-market (or bank) account:

$$\begin{array}{lcl} N_t & = & \exp(r_0 + \ldots + r_{t-1}) = A_{0,t}\,, \text{where} \\ (A_{0,t})^{-1} & = & \mathbb{E}_0(M_{0,1}) \cdots \mathbb{E}_{t-1}(M_{t-1,t})\,, \end{array}$$

  the associated equivalent probability $\mathbb{Q}$ has a one-period conditional density, with respect to $\mathbb{P}$, given by :

$$d_{t+1}^{\mathbb{Q}} = \frac{d\mathbb{Q}_{t,t+1}}{d\mathbb{P}_{t,t+1}} = \frac{A_{0,t+1} M_{t,t+1}}{A_{0,t}} = \frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})} = e^{r_t}\, M_{t,t+1}\,.$$

  and it is called **risk-neutral probability measure**.

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Risk Neutral Dynamics:

- In a general $(T-t)$-period horizon, the conditional (to $\underline{w_t}$) density of the risk-neutral probability $\mathbb{Q}$ with respect to the historical probability $\mathbb{P}$ is given by:

$$\begin{aligned}\frac{d\mathbb{Q}_{t,T}}{d\mathbb{P}_{t,T}} &= \frac{M_{t,t+1}\cdot\ldots\cdot M_{T-1,T}}{\mathbb{E}_t(M_{t,t+1})\cdot\ldots\cdot\mathbb{E}_{T-1}(M_{T-1,T})} \\ &= \exp(r_t+\ldots+r_{T-1})\,M_{t,T}\,.\end{aligned}$$

- This means that, for any payoff p_T at T, we have :

$$p_t = \mathbb{E}_t^{\mathbb{Q}}[\exp(-r_t-\ldots-r_{T-1})\,p_T]\,,$$

  and $p_t/A_{0,t}$ is a $\mathbb{Q}$-martingale.
- The one-period transition from the historical world to the risk-neutral one is given, in our model, by the conditional density function :

$$\frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})} = \exp\left[\Gamma_t\,\varepsilon_{t+1} - \frac{1}{2}\,\Gamma_t^2\right].$$

- Moreover, for any asset, the price p_t at t is equal to $\exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(p_{t+1})$ and, therefore, the risk premium λ_t presented above is equal to:

$$\lambda_t = \log\mathbb{E}_t(p_{t+1}) - \log\mathbb{E}_t^{\mathbb{Q}}(p_{t+1})\,.$$

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Risk Neutral Dynamics:

- The risk-neutral Laplace transform of w_{t+1}, conditionally to $\underline{w_t}$, is:

$$\mathbb{E}_t^{\mathbb{Q}}[\exp(uw_{t+1})] = \mathbb{E}_t\left[\frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})}\exp(uw_{t+1})\right]$$

$$\begin{aligned}
&= \mathbb{E}_t\left[\exp\left((\gamma_o+\gamma w_t)\,\varepsilon_{t+1} - \tfrac{1}{2}\,(\gamma_o+\gamma w_t)^2 + uw_{t+1}\right)\right] \\
&= \exp\left[u(\nu+\varphi w_t) - \tfrac{1}{2}(\gamma_o+\gamma w_t)^2\right] \times \mathbb{E}_t[\exp(\gamma_o+\gamma w_t+u\sigma)\varepsilon_{t+1}] \\
&= \exp\left[u[(\nu+\sigma\gamma_o)+(\varphi+\sigma\gamma)w_t] + \tfrac{1}{2}u^2\sigma^2\right] \\
&= \exp\left[u(\nu^*+\varphi^* w_t) + \tfrac{1}{2}u^2\sigma^2\right],
\end{aligned}$$

Prop. I.3

Under the risk-neutral probability $\mathbb{Q}$, w_{t+1} is an AR(1) process of the following type:

$$w_{t+1} = \nu^* + \varphi^*\, w_t \;+\; \sigma^*\eta_{t+1}\,,$$

with

$$\nu^* = (\nu+\sigma\gamma_o)\,,\;\; \varphi^* = (\varphi+\sigma\gamma)\,,\;\; \sigma^* = \sigma\,.$$

and where $\eta_{t+1} \overset{\mathbb{Q}}{\sim} \mathcal{IIN}(0,1)$. Note that $\varepsilon_{t+1} = \eta_{t+1} + \Gamma_t$.

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Risk Neutral Dynamics:

- The exponential-affine SDF-based change of probability measure preserves the Gaussian AR (affine) nature of (w_t) under $\mathbb{Q}$. Parameters ν and ρ change, while σ remains the same.
- If $\Gamma_t = \gamma_o$ (constant market price of risk), only the constant term changes. If $\Gamma_t = 0$, then (w_t) has the same distribution under $\mathbb{P}$ and $\mathbb{Q}$.
- Indeed, if $\Gamma_t = 0$ we have $M_{t,t+1} = \exp(-r_t)$ and any payoff is discounted under $\mathbb{P}$ by the risk-free rate:

$$B(t,h) = \mathbb{E}_t[\exp(-r_t - \ldots - r_{t+h-1})]\,, \quad \text{with} \quad r_t = \beta + \alpha w_t\,.$$

$M_{t,t+1} = \exp(-r_t)$ implies that we do not consider the factor (w_t) as a source of risk affecting the ZCB price process, and in that case we have $\lambda_t = 0$.

Remark I.2

If $w_t = r_t$, the AAO condition $\mathbb{E}_t(M_{t,t+1}) = \exp(-r_t)$ implies $\beta = 0$ and $\alpha = 1$. We have to guarantee that the theoretical formula $R(t,h)$ generates, when $h = 1$, exactly the short rate process we have assumed under $\mathbb{P}$. It is the discrete-time equivalent of the continuous-time Vasicek (1977) model.

Remark I.3

Simple generalization to the Gaussian AR(p) case: $W_t = (w_t, \ldots, w_{t-p+1})'$ instead of w_t.

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Yield curve shapes:

- ☐ Which kind of yield curve shapes are we able to generate thanks to the introduction of lagged factor values ?

- Compared to the Gaussian AR(1) case, are we able to generate yield curves closer to the observed ones ?

- Let us consider (from CRSP) a data set on the U. S. term structure of interest rates (treasury zero-coupon bond yields), covering the period from June 1964 to December 1995. We have 379 monthly observations for each of the nine maturities : 1, 3, 6 and 9 months and 1, 2, 3, 4 and 5 years.

- The term structure of ZCB yields is, on average:
  - *i*) upward sloping
  - *ii*) and the yields with larger standard deviation, positive skewness and kurtosis are those with shorter maturities.
  - *iii*) Moreover, yields are **highly autocorrelated** with a persistence which is increasing with the time to maturity.
- Let us take as factor the 1-month yield : $r_t = R(t, 1\,month)$.

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Yield curve shapes:

Table 1 : Summary Statistics on U. S. Monthly Yields from June 1964 to December 1995. ACF(k) indicates the empirical autocorrelation between yields $R(t, h)$ and $R(t-k, h)$, with h and k expressed on a monthly basis.

Maturity	1-m	3-m	6-m	9-m	1-yr	2-yr	3-yr	4-yr	5-yr
Mean	0.0645	0.0672	0.0694	0.0709	0.0713	0.0734	0.0750	0.0762	0.0769
Std.Dev.	0.0265	0.0271	0.0270	0.0269	0.0260	0.0252	0.0244	0.0240	0.0237
Skew.	1.2111	1.2118	1.1518	1.1013	1.0307	0.9778	0.9615	0.9263	0.8791
Kurt.	4.5902	4.5237	4.3147	4.1605	3.9098	3.6612	3.5897	3.5063	3.3531
Min	0.0265	0.0277	0.0287	0.0299	0.0311	0.0366	0.0387	0.0397	0.0398
Max	0.1640	0.1612	0.1655	0.1644	0.1581	0.1564	0.1556	0.1582	0.1500
ACF(1)	0.9587	0.9731	0.9747	0.9745	0.9727	0.9780	0.9797	0.9802	0.9822
ACF(5)	0.8288	0.8531	0.8579	0.8588	0.8604	0.8783	0.8915	0.8986	0.9053
ACF(10)	0.7278	0.7590	0.7691	0.7699	0.7683	0.7885	0.8021	0.8075	0.8212
ACF(20)	0.4303	0.4631	0.4880	0.4996	0.5156	0.5742	0.6051	0.6193	0.6431
ACF(30)	0.2548	0.2682	0.3016	0.3213	0.3518	0.4358	0.4725	0.4994	0.5187
ACF(40)	0.1362	0.1415	0.1677	0.1853	0.2160	0.3056	0.3427	0.3780	0.3961

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Yield curve shapes:

- **Figures A, B, C and D:** examples of observed yield curves in the data base.
- **Figures from 1 to 4:** yield curves generated by a Gaussian AR(1) ATSM.
  $\hookrightarrow$ Shapes **can be only** monotone increasing/decreasing, flat or with hump.
- **Figures from 5 to 8:** yield curves generated by a Gaussian AR(2) ATSM.
  $\hookrightarrow$ Richer but not really realistic shapes.
- **Figures from 9 to 12:** yield curves generated by a Gaussian AR(3) ATSM.
  $\hookrightarrow$ Richer and more realistic shapes (two humps).

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Yield curve shapes:

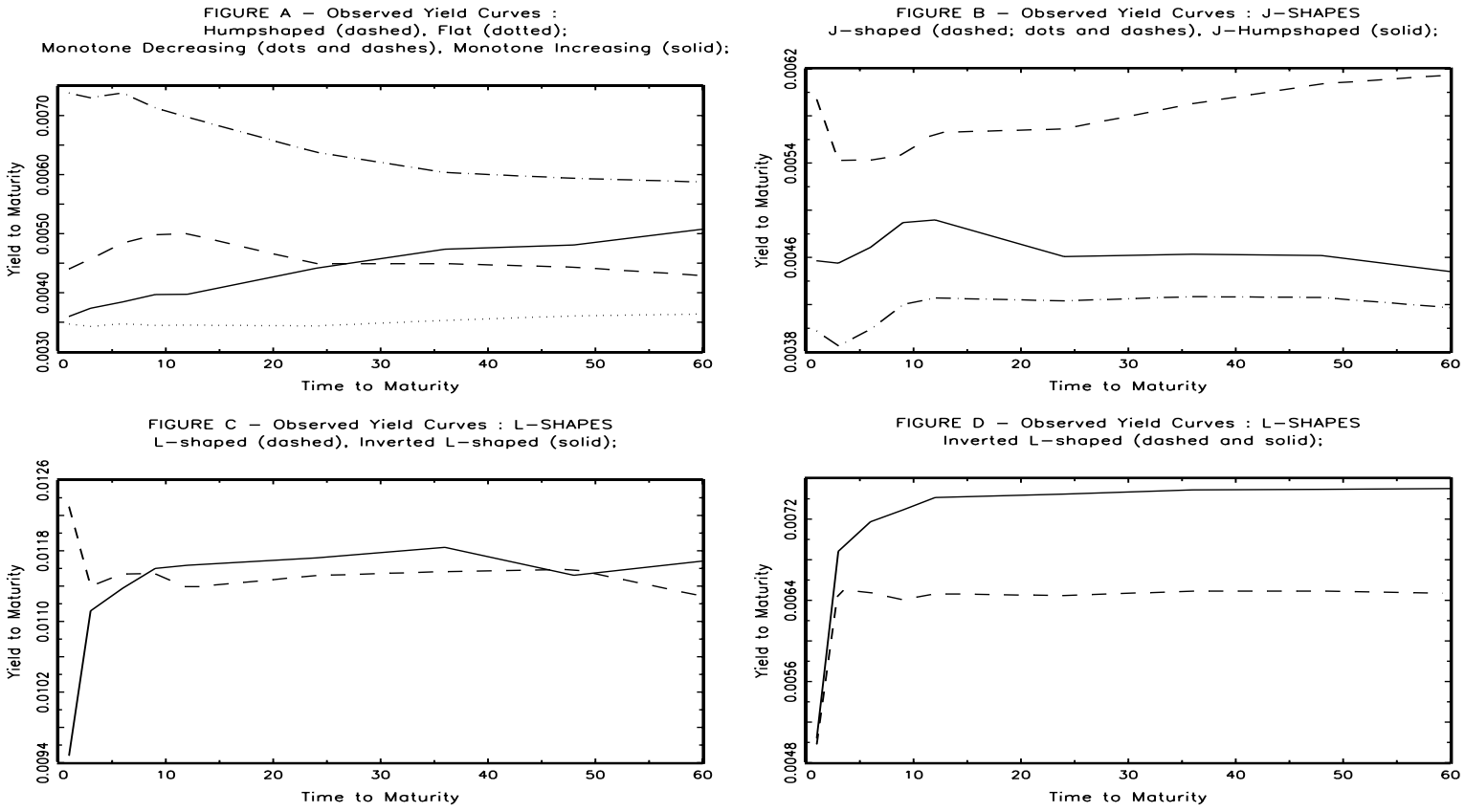

Figure 1: Observed Term Structure Shapes.

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Yield curve shapes:

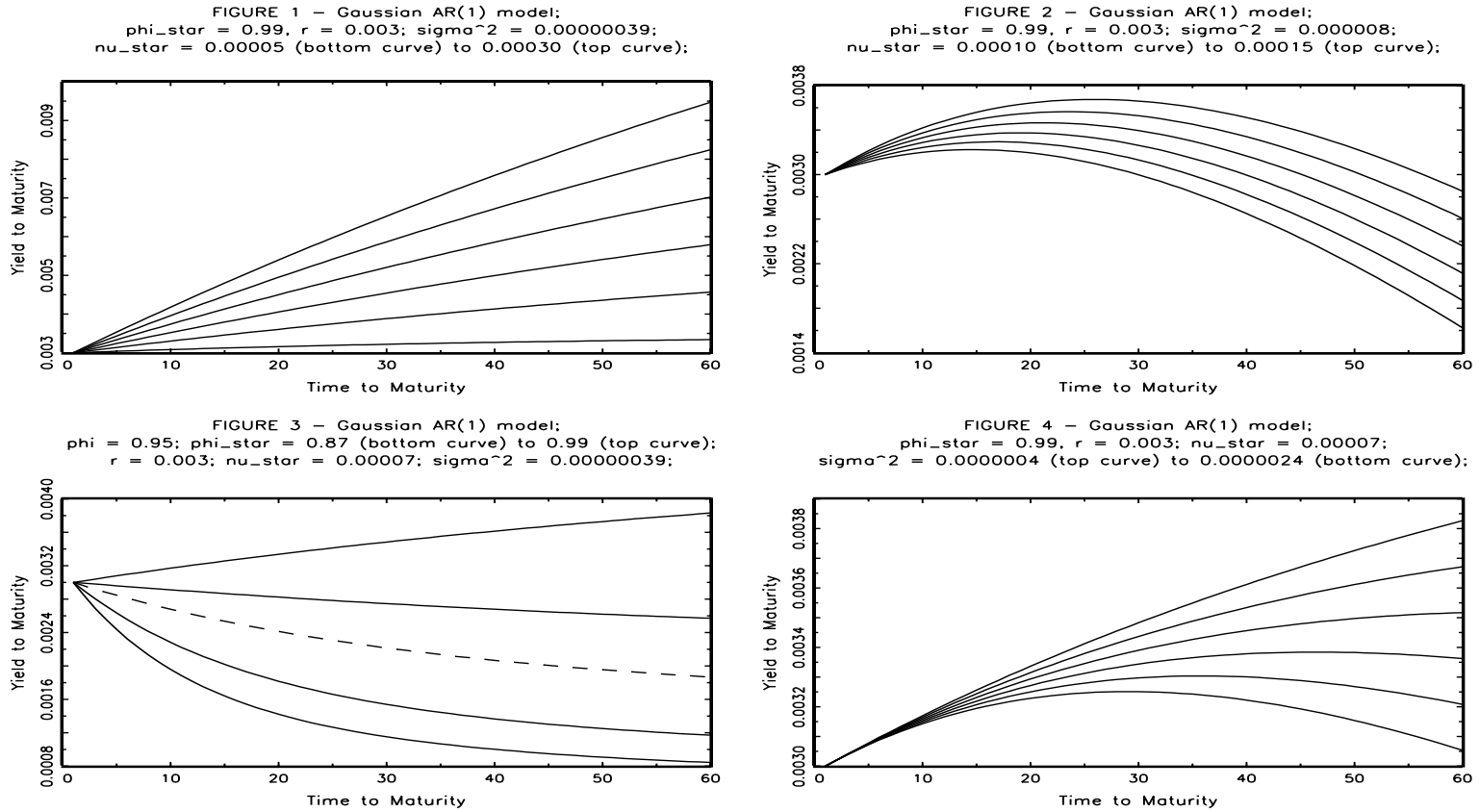

Figure 2: AR(1) possible term structure shapes.

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Yield curve shapes:

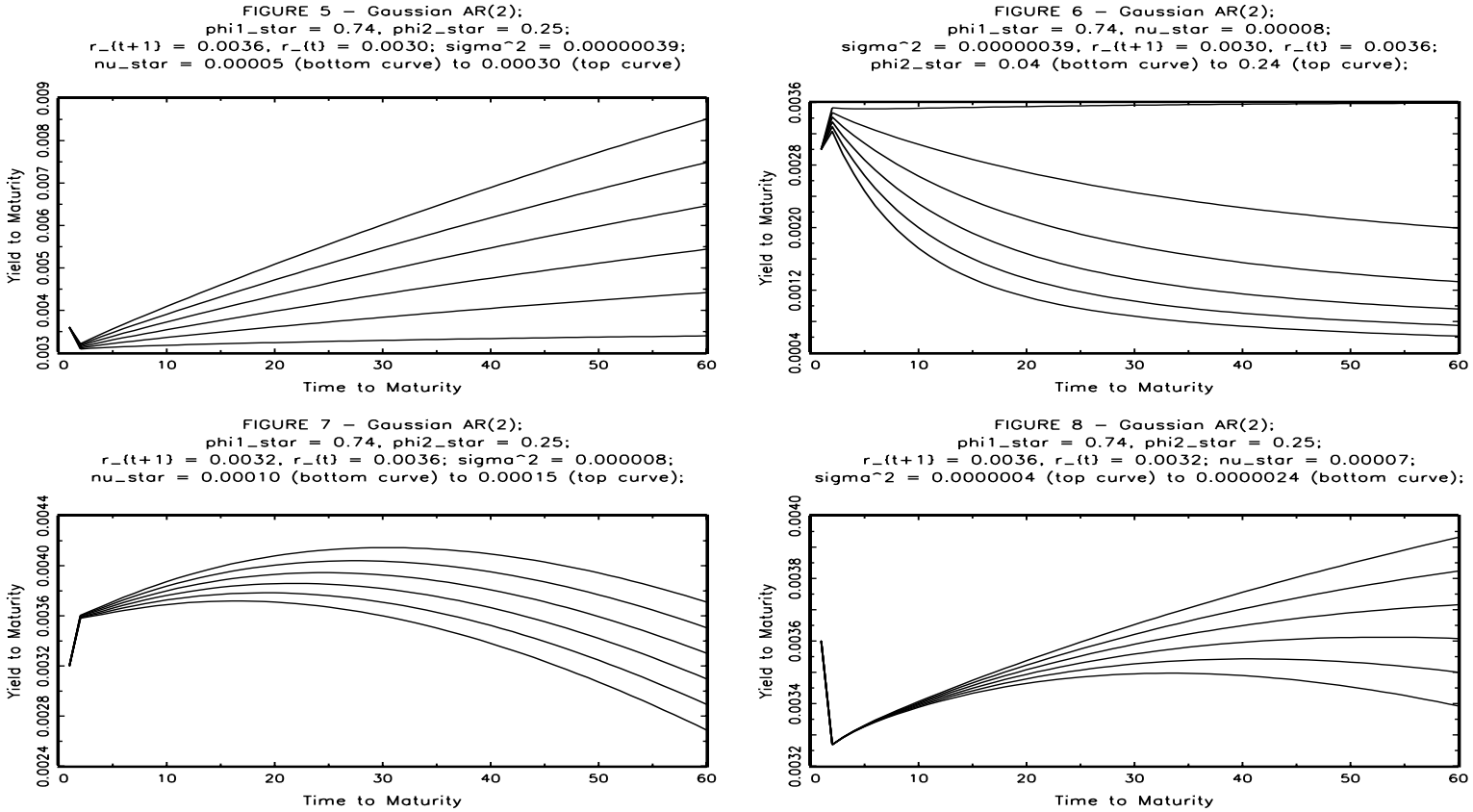

Figure 3: AR(2) possible term structure shapes.

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Yield curve shapes:

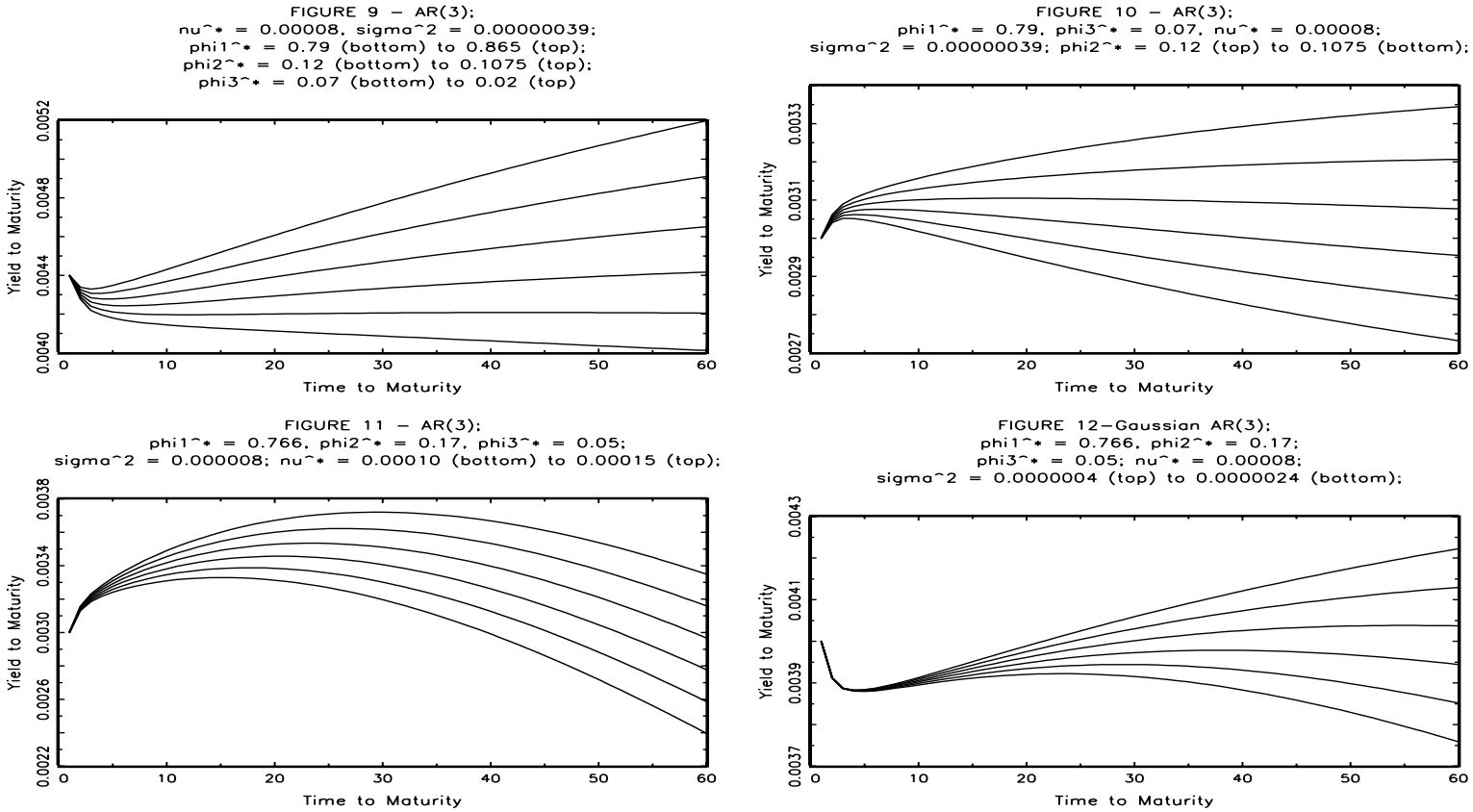

Figure 4: AR(3) possible term structure shapes.

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Yield curve shapes:

- We have to keep in mind that the shapes we have seen have been generated by **chosen** (*and not estimated* !) **parameter values**!

- **If we want to realistically verify** the ability of Gaussian AR(p) ATSMs models to generate yield curves closer to the observed one, we have to:

a) **first**, estimate the parameters of the model

b) **second**, generate the yield curves by means of the yield curve formula, fixing parameter values to their estimated values.

c) **third**, compare them with other possible (competing) yield curve models: which model fit the observed yield curves better (i.e. smallest pricing errors) ?

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Yield curve shapes (using estimated parameters):

- **Figures from 1 to 4 (slide 36):** AR(1) model-implied yield curve shapes.
- **Figures from 1 to 6 (slide 37):** AR(3) model-implied yield curve shapes.
- **Figures from 1 to 6 (slide 38):** AR(4) model-implied yield curve shapes.
- **Figures from 1 to 6 (slide 39):** AR(5) model-implied yield curve shapes.
- **Figures from 1 to 6 (slide 40):** AR(6) model-implied yield curve shapes.

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Yield curve shapes (using estimated parameters):

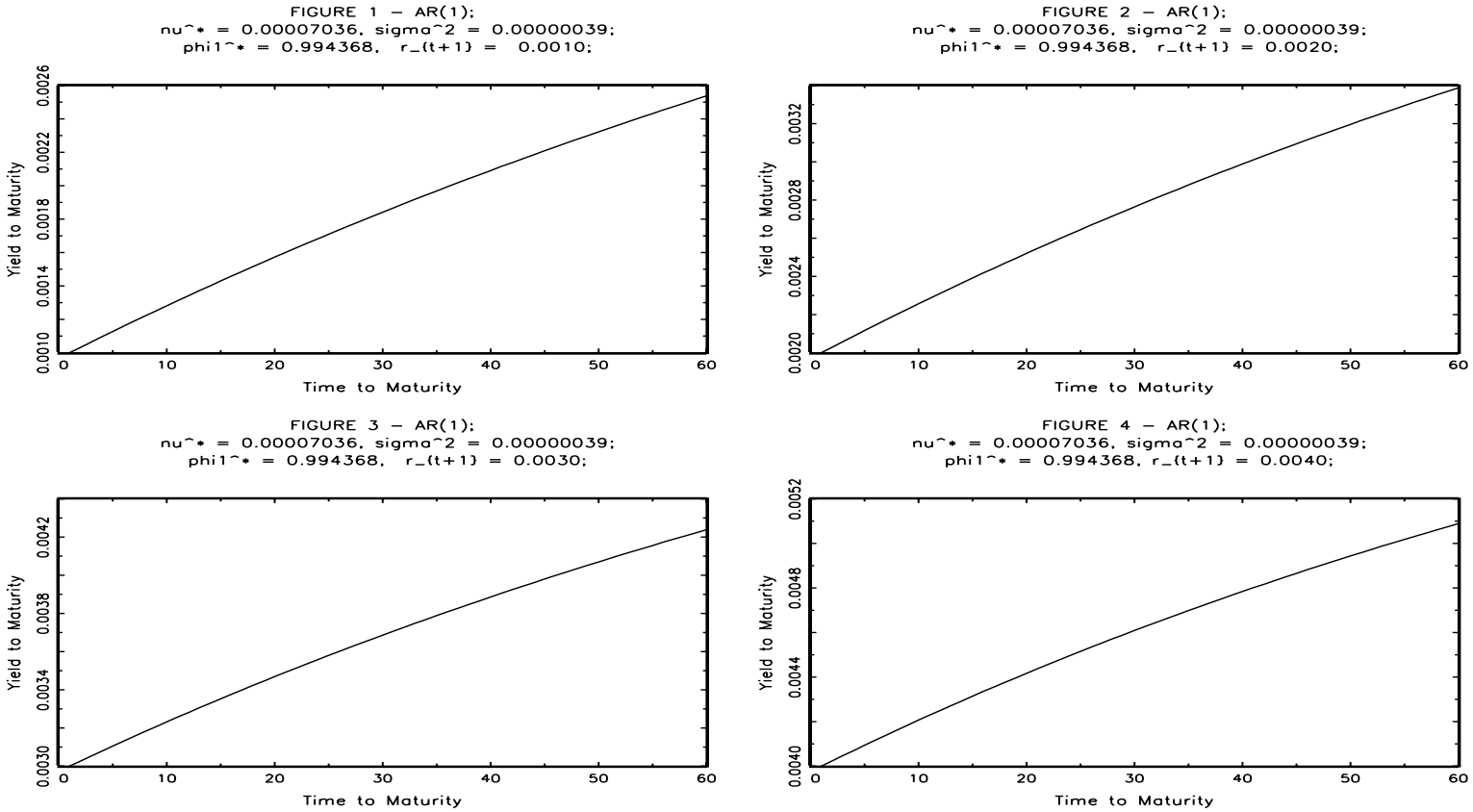

Figure 5: AR(1) estimated term structure shapes.

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Yield curve shapes (using estimated parameters):

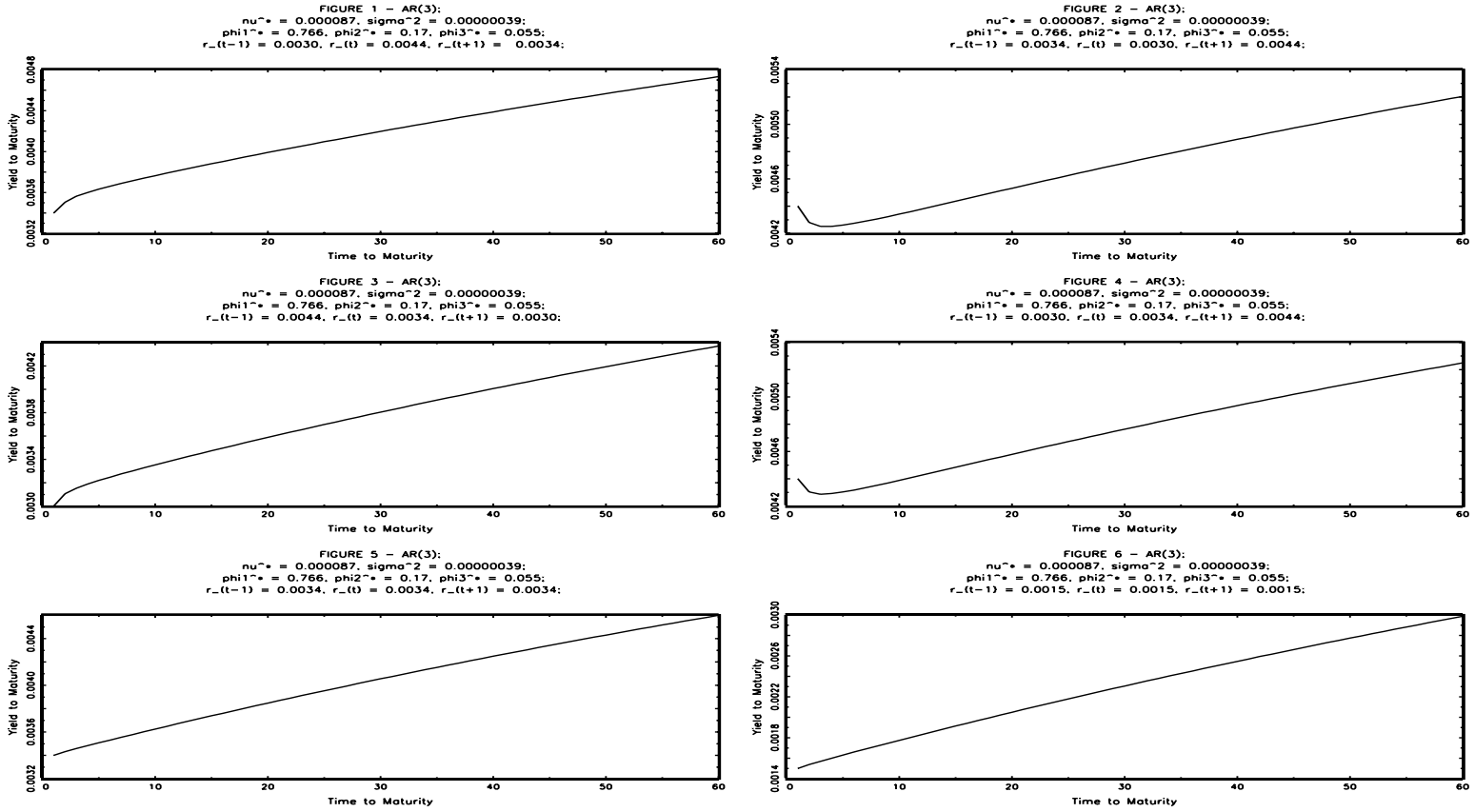

Figure 6: AR(3) estimated term structure shapes.

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Yield curve shapes (using estimated parameters):

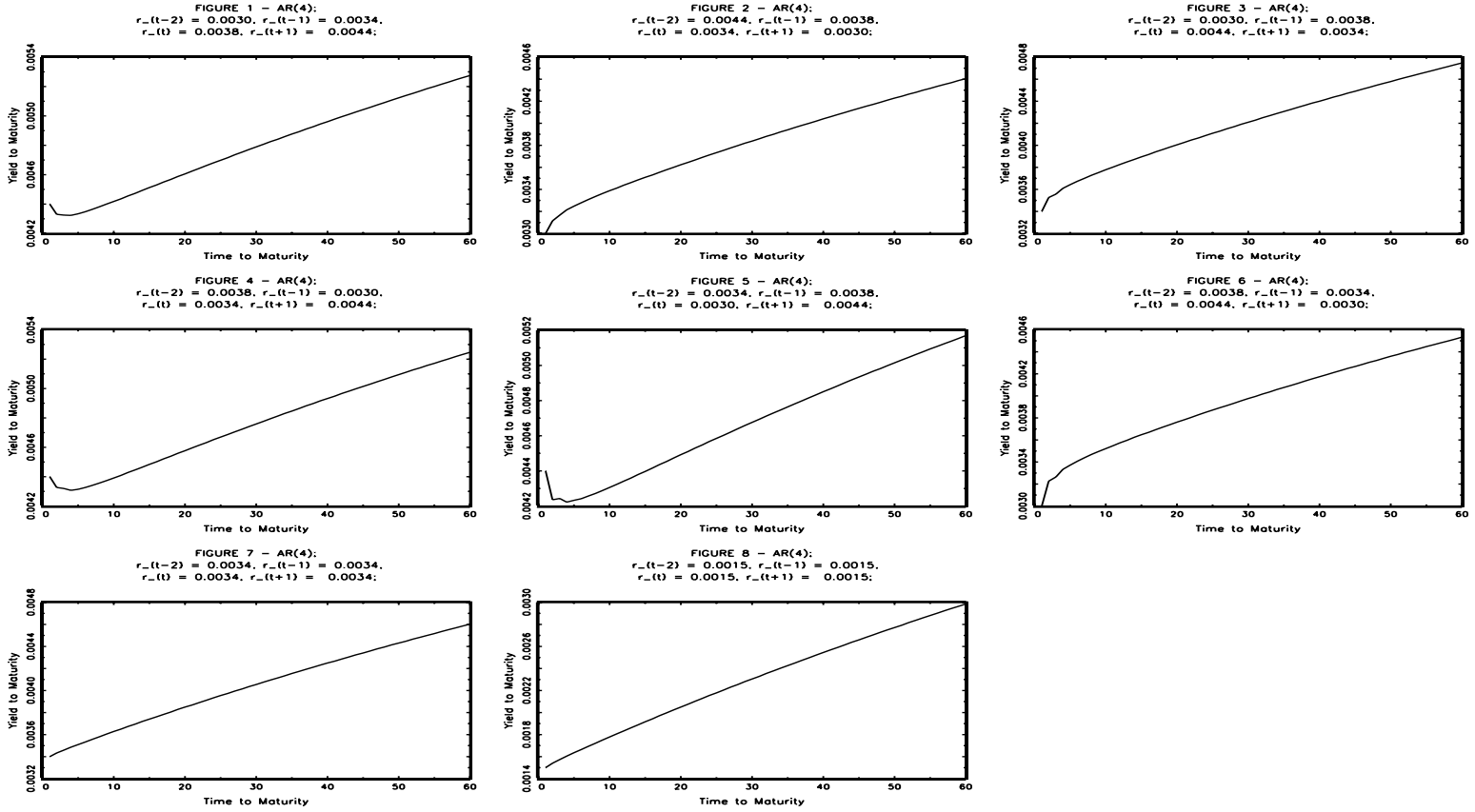

Figure 7: AR(4) estimated term structure shapes.

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Yield curve shapes (using estimated parameters):

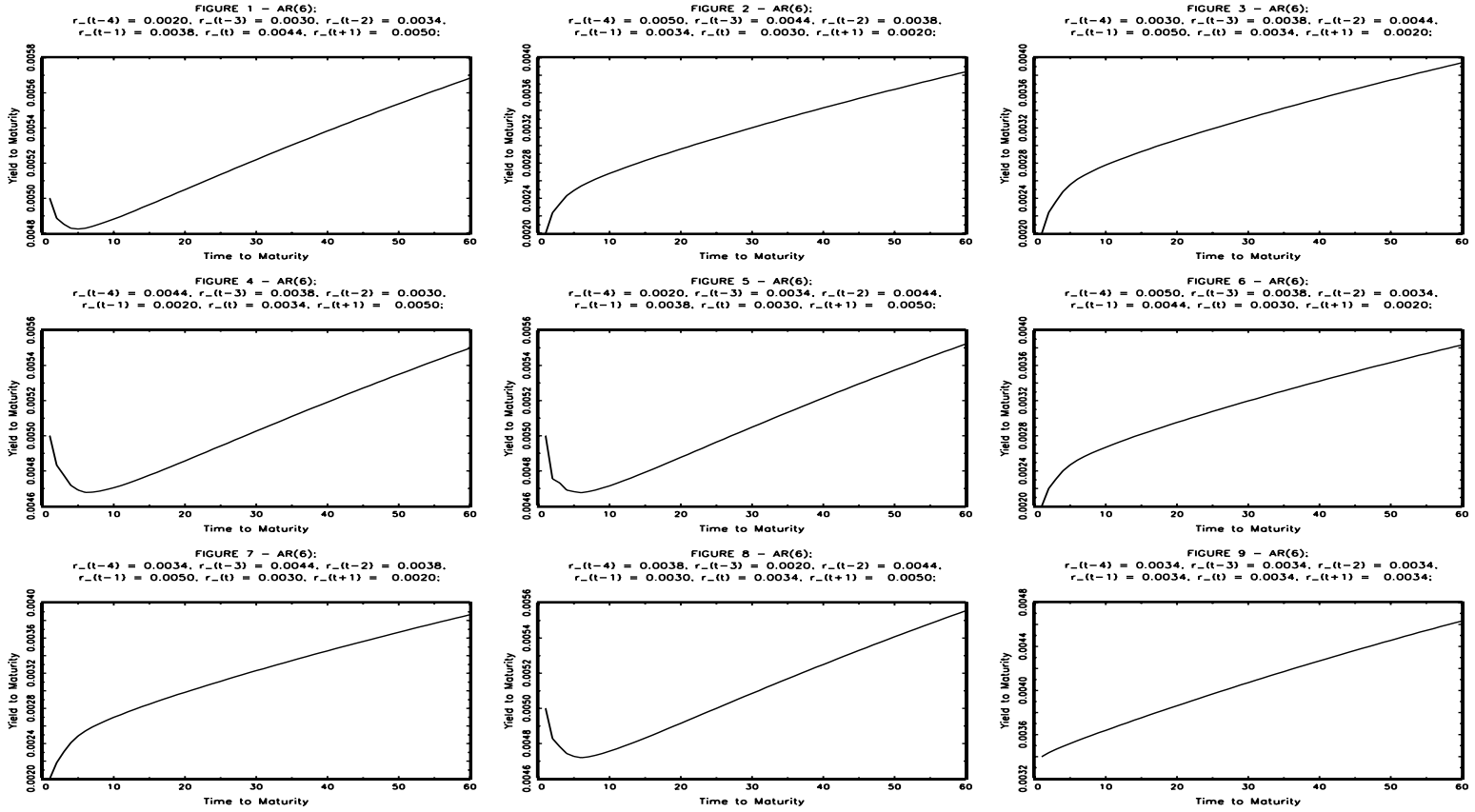

Figure 8: AR(6) estimated term structure shapes.

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I.4 – BIVARIATE GAUSSIAN ATSMs

Historical Dynamics:

- We consider again our discrete-time economy between dates 0 and T; w_t is our **factor or a state vector**, and it may be observable, partially observable or unobservable by the econometrician.
- The Gaussian AR(p) ATSM is (at estimated parameters) not able to completely explain the variability over time and maturities of the observed yield curves $\Rightarrow$ we need more information, i.e. more factors!
- The size of $w_t = (w_{1,t}, w_{2,t})'$ is now assumed to be $K = 2$. The **historical dynamics** of w_t is defined by the joint distribution of $\underline{w}_T = (w_0, \ldots, w_T)$, denoted by $\mathbb{P}$, or by the conditional probability density function (p.d.f.):

$$f_t(w_{1,t+1}, w_{2,t+1} | \underline{w}_t) \,,$$

or by the **conditional Laplace transform** (L.T.):

$$\varphi_t(u|\underline{w}_t) = \mathbb{E}[\exp(u_1\, w_{1,t+1} + u_2\, w_{2,t+1}) \,|\, \underline{w}_t] = \mathbb{E}_t[\exp(u' w_{t+1})] \,.$$

- We also introduce the **conditional Log-Laplace transform**:

$$\psi_t(u|\underline{w}_t) = \psi_t(u) = \mathrm{Log}[\varphi_t(u|\underline{w}_t)] \,.$$

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I.4 – BIVARIATE GAUSSIAN ATSMs

Historical Dynamics:

- The 2-dimensional factor $w_{t+1} = (w_{1,t+1}, w_{2,t+1})'$ is a Gaussian VAR(1) process:

$$w_{t+1} \quad = \quad \nu + \Phi w_t + \Sigma \varepsilon_{t+1} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} + \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} w_t + \Sigma \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

  where ε_t is a 2-dimensional Gaussian white noise with $\mathcal{N}(0, I_2)$ distribution.

- $\mathbb{E}_t[w_{t+1}] = \nu + \Phi w_t$ and $\mathbb{V}ar_t[w_{t+1}] = \Sigma\Sigma' = \Omega$.

  $(w_{t+1} \mid w_t) \sim N_2(\nu + \Phi w_t, \Omega)$ (under $\mathbb{P}$).

- At date t, the k-step ahead forecast (denoted $w^e_{t+k \mid t}$) is:

$$w^e_{t+k \mid t} := \mathbb{E}_t[w_{t+k}] = (I_2 + \Phi + \ldots + \Phi^{k-1})\, \nu + \Phi^k\, w_t\,.$$

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I.4 – BIVARIATE GAUSSIAN ATSMs

Historical Dynamics:

- The Laplace transform of a 2-dimensional Gaussian random variable $Y \sim N_2(\mu, \Xi)$, with $\mu = (\mu_1, \mu_2)'$, $\Xi_{11} = V[Y_1]$, $\Xi_{22} = V[Y_2]$, $\Xi_{12} = Cov[Y_1, Y_2] = \Xi_{21}$, is:

$$\varphi(u) = E[\exp(u_1 Y_1 + u_2 Y_2)] = \exp\left(u'\mu + \frac{1}{2}\, u'\Xi u\right)$$

$$= \quad \exp\left[(u_1\mu_1 + u_2\mu_2) + \frac{1}{2}\,(u_1^2\, V[Y_1] + u_2^2\, V[Y_2] + 2\, u_1\, u_2 Cov[Y_1, Y_2])\right]$$

- This means that:

$$\begin{aligned}\varphi_t(u|\underline{w}_t) &= \varphi_t(u) = \exp\left[u'(\nu + \Phi w_t) + \frac{1}{2}\, u'\,\Omega u\right] \\ &= \exp\left[(u'\,\nu + \frac{1}{2}\, u'\,\Omega u) + u'\,\Phi w_t\right],\end{aligned}$$

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I.4 – BIVARIATE GAUSSIAN ATSMs

The Stochastic Discount Factor:

- We specify the following SDF:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' w_t + \Gamma_t' \varepsilon_{t+1} - \frac{1}{2}\Gamma_t'\Gamma_t\right],$$

the coefficients $\alpha = [\alpha_1, \alpha_2]'$ and β are path independent, and $\Gamma_t = (\gamma_o + \gamma w_t)$, where $\gamma_o = (\gamma_{1,o}, \gamma_{2,o})'$ and γ is a $(2,2)$-matrix:

$$\begin{aligned}
\Gamma_{1,t} &= \gamma_{o,1} + \gamma_{1,1}\, w_{1,t} + \gamma_{1,2}\, w_{2,t} = \gamma_{1,o} + \widetilde{\gamma}_1' w_t \\
\Gamma_{2,t} &= \gamma_{o,2} + \gamma_{2,1}\, w_{1,t} + \gamma_{2,2}\, w_{2,t} = \gamma_{2,o} + \widetilde{\gamma}_2' w_t\,.
\end{aligned}$$

- This means that, at any date t, the risk-correction coefficients associated to the first and second factor, i.e. $\Gamma_{1,t}$ and $\Gamma_{2,t}$ respectively, are a linear combination of BOTH scalar factors $w_{1,t}$ and $w_{2,t}$.

- The no-arbitrage restriction $\mathbb{E}_t(M_{t,t+1}) = \exp(-r_t)$, implies the relation $r_t = \beta + \alpha' w_t = \beta + \alpha_1\, w_{1,t} + \alpha_2\, w_{2,t}$.

- For a given payoff $\exp(-b' w_{t+1})$ at $t+1$, the risk premium is now $\lambda_t = b'\,\Sigma\,\Gamma_t = b'\,\Sigma\,(\gamma_o + \gamma\, w_t)$.

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I.4 – BIVARIATE GAUSSIAN ATSMs

The Affine Yield Curve Formula:

- The price at date t of the zero-coupon bond with time to maturity h is :

$$B(t,h) = \exp(C_h' w_t + D_h) = \exp(C_{1,h} w_{1,t} + C_{2,h} w_{2,t} + D_h)\,, \quad h \geq 1\,,$$

where c_h and d_h satisfies, for $h \geq 1$, the recursive equations:

$$\begin{cases} C_h &= -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} \\ &= -\alpha + \Phi^{*'} C_{h-1}\,, \\ D_h &= -\beta + C_{h-1}'(\nu + \Sigma\gamma_o) + \frac{1}{2} C_{h-1}'(\Sigma\Sigma') C_{h-1} + D_{h-1} \\ &= -\beta + C_{h-1}'\nu^* + \frac{1}{2} C_{h-1}'\Omega C_{h-1} + D_{h-1}\,, \end{cases}$$

with initial conditions $C_0 = 0, D_0 = 0$ (or $C_1 = -\alpha, D_1 = -\beta$). The affine term structure of interest rates formula is:

$$\begin{aligned} R(t,h) &= -\frac{1}{h}\log B(t,h) = -\frac{C_h'}{h} w_t - \frac{D_h}{h} \\ &= -\frac{1}{h}\left(C_{1,h}\, w_{1,t} + C_{2,h}\, w_{2,t} + D_h\right), \quad h \geq 1\,, \end{aligned}$$

and for a given date t, any yield $R(t,h)$ is an affine function of the 2-dimensional factor $w_t = (w_{1,t}, w_{2,t})'$.

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I.4 – BIVARIATE GAUSSIAN ATSMs

Risk Neutral Dynamics:

Prop. I.4

Under the risk-neutral probability $\mathbb{Q}$, w_{t+1} is a bivariate VAR(1) process of the following type:

$$w_{t+1} = \nu^* + \Phi^* w_t + \Sigma^* \varepsilon_{t+1} = \begin{bmatrix} \nu_1^* \\ \nu_2^* \end{bmatrix} + \begin{bmatrix} \varphi_{11}^* & \varphi_{12}^* \\ \varphi_{21}^* & \varphi_{22}^* \end{bmatrix} w_t + \Sigma^* \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

with

$$\nu^* = (\nu + \Sigma \gamma_o), \quad \Phi^* = (\Phi + \Sigma \gamma), \quad \Sigma^* = \Sigma.$$

and where $\eta_{t+1} \overset{\mathbb{Q}}{\sim} \mathcal{IIN}(0, I_2)$. Note that $\varepsilon_{t+1} = \eta_{t+1} + \Gamma_t$.

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I.4 – BIVARIATE GAUSSIAN ATSMs

The Observable Factor Case:

- The 2-dimensional factor (w_t) can be considered as a vector of two yields: the first component is assumed to be the short rate r_t and the second one is the long rate R_t.
- More precisely, we assume:

$$w_t = \begin{bmatrix} R(t,1) \\ R(t,H) \end{bmatrix}$$

where $R(t,1) = r_t$ and $R(t,H) = R_t$.

- We can start better understanding the role of the no-arbitrage restrictions and ICCs. First, I have to impose that $R(t,1) = r_t$. This condition generates the AAO restriction:

$$R(t,1) = \beta + \alpha' w_t = \beta + \alpha_1 r_t + \alpha_2 R_t = r_t$$

$$\Leftrightarrow \beta = 0\,, \quad \alpha_1 = 1\,, \quad \alpha_2 = 0\,,$$

These conditions are equivalent to $C_1 = -(1,0)$ and $D_1 = 0$ and they are analytically satisfied.

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I.4 – BIVARIATE GAUSSIAN ATSMs

The Observable Factor Case:

- Second, I have to impose that $R(t,H) = R_t$ for any t. In this case we have:

$$-\frac{1}{H}[C_{1,H}\, r_t + C_{2,H}\, R_t + D_H] = R_t$$

$$\Leftrightarrow C_{1,H}\, r_t + C_{2,H}\, R_t + D_H = -HR_t$$

$$\Leftrightarrow C_{1,H} = 0\,, \quad C_{2,H} = -H\,, \quad D_H = 0\,,$$

that is $C_H = -H\,(0,1)'$ and $D_H = 0$. In this case, the absence of arbitrage conditions for the 2 yields in w_t imply :

$$\begin{aligned} (i)\ C_1 &= -(1,0)'\,, \quad D_1 = 0\,, \\ (ii)\ C_H &= -H\,(0,1)'\,, \quad D_H = 0\,. \end{aligned}$$

- The first set of conditions is used as initial value in the recursive equations (C_h, D_h).
- The second condition imply restrictions on model parameters which must be taken into account at the estimation stage. We have to impose to the yield-to-maturity formula to pass through the yields in w_t.

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I.5 – MULTIVARIATE GAUSSIAN ATSMs

- Let us assume that the K-dimensional factor $w_{t+1} = (w_{1,t+1}, \ldots, w_{K,t+1})'$ is a Gaussian VAR(1) process of the following type:

$$w_{t+1} \quad = \quad \nu + \Phi w_t + \Sigma \varepsilon_{t+1} \,,$$

where $\varepsilon_{t+1} = (\varepsilon_{1,t+1}, \ldots, \varepsilon_{K,t+1})$ is a K-dimensional Gaussian white noise with $\mathcal{N}(0, I_K)$ distribution. We thus have:

$$\mathbb{E}_t[\exp(u' w_{t+1})] = \exp\left[u' \mathbb{E}_t[w_{t+1}] + \frac{1}{2} u' \, \mathbb{V}ar[w_{t+1}] u\right] .$$

- We specify the following SDF:

$$M_{t,t+1} = \exp\left[-\beta - \alpha' w_t + \Gamma_t' \varepsilon_{t+1} - \frac{1}{2} \Gamma_t' \Gamma_t\right] ,$$

- $\Gamma_t = (\gamma_o + \gamma w_t)$, where $\gamma_o = (\gamma_{1,o}, \ldots, \gamma_{K,o})'$ and γ is a (K, K)-matrix:

$$\begin{aligned}
\Gamma_{1,t} &= \gamma_{1,o} + \gamma_{1,1} \, w_{1,t} + \gamma_{1,2} \, w_{2,t} + \ldots + \gamma_{1,K} \, w_{K,t} = \gamma_{1,o} + \widetilde{\gamma}_1' w_t \\
&\vdots \\
\Gamma_{K,t} &= \gamma_{K,o} + \gamma_{K,1} \, w_{1,t} + \gamma_{K,2} \, w_{2,t} + \ldots + \gamma_{K,K} \, w_{K,t} = \gamma_{K,o} + \widetilde{\gamma}_K' w_t \,.
\end{aligned}$$

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I.5 – MULTIVARIATE GAUSSIAN ATSMs

- This means that, at any date t, the risk-correction coefficient associated to the jth factor, i.e. $\Gamma_{j,t}$, is a linear combination of ALL the K scalar factors.

- The condition $\mathbb{E}_t(M_{t,t+1}) = \exp(-r_t)$, implies $r_t = \beta + \alpha' w_t$, and the risk premium is $\lambda_t = b' \Sigma \Gamma_t = b' \Sigma (\gamma_o + \gamma w_t)$.

- The price at date t of the zero-coupon bond with time to maturity h is :

$$B(t,h) = \exp(C_h' w_t + D_h) = \exp(C_{1,h} w_{1,t} + \ldots + C_{K,h} w_{K,t} + D_h)\,, \quad h \geq 1\,,$$

where c_h and d_h satisfies, for $h \geq 1$, the recursive equations:

$$\begin{cases} C_h &= -\alpha + (\Phi + \Sigma\gamma)' C_{h-1} = -\alpha + \Phi^{*'} C_{h-1}\,, \\ D_h &= -\beta + C_{h-1}'(\nu + \Sigma\gamma_o) + \frac{1}{2} C_{h-1}'(\Sigma\Sigma') C_{h-1} + D_{h-1} \\ &= -\beta + C_{h-1}'\nu^* + \frac{1}{2} C_{h-1}' \Omega C_{h-1} + D_{h-1}\,, \end{cases}$$

with initial conditions $C_0 = 0, D_0 = 0$ (or $C_1 = -\alpha, D_1 = -\beta$).

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I.5 – MULTIVARIATE GAUSSIAN ATSMs

- Under no-arbitrage, and for a fixed maturity T, the one-period geometric zero-coupon bond return process $\rho = [\,\rho(t,T), 0 \leq t \leq T\,]$, where $\rho(t+1,T) = \log[B(t+1,\ T)] - \log[B(t,\ T)]$, is given by:

$$\begin{aligned}\rho(t+1,\ T) \quad &= \quad r_t - \tfrac{1}{2}\,\omega(t+1,\ T)'\omega(t+1,\ T) + \omega(t+1,\ T)'\Gamma_t \\ &\quad -\omega(t+1,\ T)'\,\varepsilon_{t+1}\,,\end{aligned}$$

where $\omega(t+1,\ T) = -(\Sigma' C_{T-t-1})$ is an K-dimensional vector. The associated risk premium, between t and $t+1$, is given by :

$$\lambda_t(T) = \omega(t+1,\ T)'\,\Gamma_t = \sum_{i=1}^{K} \omega_i(t+1,\ T)\,\Gamma_{i,t}\,,$$

where $\omega(t+1,\ T) = [\omega_1(t+1,\ T), \ldots, \omega_K(t+1,\ T)]'$.

- The magnitude of $\lambda_t(T)$ is given by a linear combination of the K scalar risk premia $\Gamma_{i,t} = \gamma_{o,i} + \widetilde{\gamma}_i' w_t$. In other words, ALL scalar factors $w_{i,t}$, with $i \in \{1, \ldots, K\}$, determine the magnitude and the variability over time of ANY (scalar) risk premia $\Gamma_{i,t}$.

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I.5 – MULTIVARIATE GAUSSIAN ATSMs

- The affine term structure of interest rates is given by:

$$R(t,h) = -\frac{1}{h}\log B(t,h) = -\frac{C_h'}{h}w_t - \frac{D_h}{h}\,, \quad h \geq 1\,,$$

- The risk-neutral Laplace transform of w_{t+1}, conditionally to $\underline{w_t}$, is:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}[\exp(u'w_{t+1})] &= \mathbb{E}_t\left[\frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})}\exp(u'w_{t+1})\right] \\
&= \mathbb{E}_t\left[\exp\left(\Gamma_t'\,\varepsilon_{t+1} - \tfrac{1}{2}\,\Gamma_t'\Gamma_t + u'w_{t+1}\right)\right] \\
&= \exp\left[u'(\nu + \Phi w_t) - \tfrac{1}{2}\Gamma_t'\Gamma_t)\right]\,\mathbb{E}_t[\exp(\Gamma_t + \Sigma' u)'\varepsilon_{t+1}] \\
&= \exp\left[u'[(\nu + \Sigma\,\gamma_o) + (\Phi + \Sigma\,\gamma)w_t] + \tfrac{1}{2}u'(\Sigma\Sigma')u\right]\,.
\end{aligned}$$

- Under $\mathbb{Q}$, w_{t+1} is the following K-dimensional VAR(1) process:

$$w_{t+1} = \nu^* + \Phi^* w_t + \Sigma^*\eta_{t+1}\,,$$

with

$$\nu^* = (\nu + \sigma\gamma_o)\,,\;\; \Phi^* = (\Phi + \Sigma\gamma)\,,\;\; \Sigma^* = \Sigma\,,$$

and where η_{t+1} is (under $\mathbb{Q}$) an K-dimensional Gaussian white noise with $\mathcal{N}(0, I_K)$ distribution.

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I.5 – MULTIVARIATE GAUSSIAN ATSMs

- The K-dimensional factor (w_t) can be considered as a vector of yields at different maturities in which the first component is assumed to be the short rate r_t. More precisely, we assume:

$$w_t = \begin{bmatrix} R(t, h_1) \\ R(t, h_2) \\ \vdots \\ R(t, h_K) \end{bmatrix}$$

where $R(t, h_1) = R(t, 1) = r_t$ and $h_1 < h_2 < \ldots < h_K$.

- The absence of arbitrage conditions for the K yields in w_t imply :

$$\begin{array}{lll} (i)\ C_1 & = & -e_1\,, \quad D_1 = 0\,, \\ (ii)\ C_{h_j} & = & -h_j\, e_{h_j}\,, \quad D_{h_j} = 0\,, \forall\, j \in \{2, \ldots, K\}\,, \end{array}$$

where e_{h_j} denotes the h_j^{th} element of the canonical basis in $\mathbb{R}^K$.

- The first set of conditions is used as initial value in the recursive equations (C_h, D_h). The second set of $(K-1)$ conditions imply restrictions on model parameters which must be taken into account at the estimation stage.
- VAR(p) case: $W_t = (w_t', \ldots, w_{t-p+1}')'$ instead of w_t.

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I.6 – BACK MODELLING OF GAUSSIAN ATSMs

- If the $\mathbb{Q}$-dynamics of w_t is Car(1) and if r_t is an affine function of w_t, the term structure of interest rates $[R(t,h), h \in \{1, ..., H\}]$ is affine in w_t.

- Indeed, if $\psi_t^{\mathbb{Q}}(u|w_t;\theta_1^*) = a^{\mathbb{Q}}(u,\theta_1^*)'w_t + b^{\mathbb{Q}}(u,\theta_1^*)$ and $r_t = \tilde{\theta}_1 + \tilde{\theta}_2' w_t$, then, the yield-to-maturity formula is given by:

$$R(t,h) = -\frac{C_h'}{h} w_t - \frac{D_h}{h}\,, \tag{I.15}$$

where

$$\begin{cases} C_h = -\tilde{\theta}_2 + a^{\mathbb{Q}}(C_{h-1}) \\ D_h = -\tilde{\theta}_1 + b^{\mathbb{Q}}(C_{h-1}) + D_{h-1} \\ C_0 = 0,\ D_0 = 0\,. \end{cases} \tag{I.16}$$

- Note that if the ith component of w_t is a rate $R(t,h_i)$, $i \in \{1, ..., K_1\}$, we must satisfy the internal consistency conditions:

$$C_{h_i} = -h_i e_i,\ D_{h_i} = 0, \quad i \in \{1, ..., K_1\}\,.$$

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I.6 – BACK MODELLING OF GAUSSIAN ATSMs

- Let us consider the (bivariate) case where $w_t = (r(t,1), r(t,2))'$. We consider the following Gaussian VAR(1) R.N. dynamics:

$$w_{t+1} = \nu + \Phi w_t + \xi_{t+1}\,, \tag{I.17}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0, \Sigma)$. In this case, the internal consistency conditions are satisfied if we impose, in (I.15) and (I.16), $\tilde{\theta}_1 = 0$, $\tilde{\theta}_2' = (1,0)$, $c_2 = -2e_2$ and $d_2 = 0$, or :

$$\begin{cases} -2e_2 = a^{\mathbb{Q}}\begin{pmatrix}-1\\0\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\,, \\ 0 = b^{\mathbb{Q}}\begin{pmatrix}-1\\0\end{pmatrix}\,, \end{cases} \tag{I.18}$$

where $a^{\mathbb{Q}}(u) = \Phi' u$ and $b^{\mathbb{Q}}(u) = u'\nu + \frac{1}{2}u'\Sigma u$. So, relation (I.18) becomes, with obvious notations:

$$\begin{bmatrix}\varphi_{11}\\ \varphi_{12}\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}\,,$$

$$\nu_1 = \tfrac{1}{2}\sigma_1^2\,.$$

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I.6 – BACK MODELLING OF GAUSSIAN ATSMs

- Once we properly take into account the ICCs, (*I*.17) must be written:

$$\begin{cases} r(t+1,1) = \frac{1}{2}\sigma_1^2 - r(t,1) + 2r(t,2) + \xi_{1,t+1} \\ r(t+1,2) = \nu_2 + \varphi_{21} r(t,1) + \varphi_{22} r(t,2) + \xi_{2,t+1}\,, \end{cases} \tag{I.19}$$

with $\xi_t \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)$. Consequently, the R.N. conditional Log-Laplace transform of w_{t+1}, compatible with the ICCs is:

$$\psi_t^{\mathbb{Q}}(u) = u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] + \frac{1}{2} u' \Sigma u\,,$$

the yield-to-maturity formula will be affine in w_t, as indicated by (I.15), and, moreover, <u>independent</u> of the specification of the factor loading α_t.

- Now, if we move back to the historical conditional Log-Laplace transform, we get:

$$\begin{aligned} \psi_t(u) &= \psi_t^{\mathbb{Q}}(u - \alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t) \\ &= u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] - u' \Sigma \alpha_t + \frac{1}{2} u' \Sigma u\,. \end{aligned}$$

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I.6 – BACK MODELLING OF GAUSSIAN ATSMs

- If we assume $\alpha_t = \gamma + \Gamma w_t$, we get:

$$\psi_t(u) = u' \left\{ \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t \right\} + \frac{1}{2} u' \Sigma u \,,$$

or, equivalently, we have the following Car(1) $\mathbb{P}$-dynamics:

$$w_{t+1} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t + \varepsilon_{t+1} \,, \quad \text{(I.20)}$$

where $\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)$ and $\varepsilon_t = \xi_t + \Sigma(\gamma + \Gamma w_t)$.

- Observe that, even if we assume α_t to be nonlinear in w_t, the interest rate formula is still affine (contrary to the Direct Modelling case of previous sections), and the historical conditional p.d.f. of non-Car factor w_t remains known in closed form.

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Data:

- The CRSP data set on the U.S. term structure of interest rates (ZCB yields), that we consider in the following application, covers the period from June 1964 to December 1995 and contains 379 monthly observations for each of the nine maturities : 1, 3, 6 and 9 months and 1, 2, 3, 4 and 5 years.

- The summary statistics about the above mentioned (annualized) yields are presented in the Table 1 we have seen before.

- The term structure of ZCB yields is, on average:

  - upward sloping,

  - and the yields with larger standard deviation, positive skewness and kurtosis are those with shorter maturities.

  - Moreover, yields are **highly autocorrelated** with a persistence which is increasing with the time to maturity.

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Estimated Models:

- In the present empirical analysis we follow an endogenous approach, given that it gives several important advantages coming from the observations we have about the factor, that is, the short rate in the scalar case, or yields at different maturities in the multivariate framework.
- First we are able to detect stylized facts giving us the possibility to justify the AR(p) model we propose for the historical dynamics of (w_t).
- Indeed, a large empirical literature on bond yields show that interest rates have an historical multi-lag dynamics [see, among the others, Hamilton (1989), Christiansen and Lund (2003), Cochrane and Piazzesi (2005)].
- Second, observations about the (Gaussian-distributed) factor lead to an exact maximum likelihood estimation (MLE) of historical parameters: in this way, we are able to test hypotheses using likelihood ratio statistics, and rank the models in terms of various information criteria.
- Finally, the difference between directly observed and estimated factor values determine model residuals that can be used to derive various diagnostic criteria.

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Estimation Method:

- The methodology we follow to estimate the parameters of the Gaussian ATSM is based on a consistent two-step procedure.
- In the first step, thanks to observations on the K-dimensional endogenous factor (w_t), we estimate by OLS the vector of parameters:

$$\Theta_{\mathbb{P}} = [\nu', vec(\Phi)', vech(\Sigma\Sigma')']' ,$$

characterizing the historical dynamics (w_t). We have:

$$\dim(\Theta_{\mathbb{P}}) = K\,(1 + K\,p) + (K\,(K+1)/2)\,.$$

In this case the MLE coincides with OLS estimator.

- Notation: $\mathbf{W} := (w_1, \ldots, w_T)$ is (K, T) matrix of observations; $B := (\nu, \Phi_1, \ldots, \Phi_p)$ is $(K, Kp+1)$ matrix of parameters;
  - $Z_t := \begin{bmatrix} 1 \\ w_t \\ w_{t-1} \\ \vdots \\ w_{t-p+1} \end{bmatrix}$, $\mathbf{Z} := (Z_0, \ldots, Z_{T-1})$ is $((Kp+1), T)$ matrix.
  - $U := (\varepsilon_1, \ldots, \varepsilon_T)$ is (K, T) matrix. We, thus, can write $\mathbf{W} = B\,\mathbf{Z} + U$.
  - $\mathbf{w} := vec(W)$ is $(TK, 1)$ vector, $\beta := vec(B)$ is $(K^2 p + K, 1)$ vector.

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Estimation Method:

- OLS estimator : $vec(\widehat{B}) = \widehat{\beta} = vec(\mathbf{W}\,\mathbf{Z}'(\mathbf{Z}\,\mathbf{Z}')^{-1})$;
- Given that $\Omega = \mathbb{E}(\varepsilon_t\,\varepsilon_t')$, we estimate this matrix by:

$$\begin{aligned}\widehat{\Omega} &= \frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon}_t\,\widehat{\varepsilon}_t' = \frac{1}{T}\,\widehat{U}\,\widehat{U}' \\ &= \frac{1}{T}\,(\mathbf{W} - \widehat{B}\,\mathbf{Z})\,(\mathbf{W} - \widehat{B}\,\mathbf{Z}) = \frac{1}{T}\,\mathbf{W}\,(I_T - \mathbf{Z}'(\mathbf{Z}\,\mathbf{Z}')^{-1}\,\mathbf{Z})\,\mathbf{W}'\,.\end{aligned}$$

- How do we select the number of lags p (VAR order selection) in the VAR(p) model ?

a) minimizing the Forecast Mean Square Error we obtain a criterion called *Final Prediction Error (FPE)*:

$$FPE(p) = \left[\frac{T + Kp + 1}{T - Kp - 1}\right]^K det(\widehat{\Omega}(p))\,;$$

b) *Akaike's Information Criterion (AIC)*: $AIC(p) = \ln det(\widehat{\Omega}(p)) + \dfrac{2\,p\,K^2}{T}$;

c) *Hannan-Quinn Criterion (HQ)*: $HQ(p) = \ln det(\widehat{\Omega}(p)) + \dfrac{2\ln\ln T}{T}\,pK^2$;

d) *Schwarz Information Criterion (SC)*: $SC(p) = \ln det(\widehat{\Omega}(p)) + \dfrac{\ln T}{T}\,pK^2$.

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Estimation Method:

- The selected AR order p is the one minimizing the criterion.
- Small sample comparisons: $p(SC) \leq p(AIC)$ if $T \geq 8$; $p(SC) \leq p(HQ)$ for all T; $p(HQ) \leq p(AIC)$ if $T \geq 16$.

- In the second step, using observations on yields with maturities different from those used in the first step and for a given estimates of $\Theta_{\mathbb{P}}$, we estimate the vector of risk-neutral parameters:

$$\Theta_{\mathbb{Q}} = [(\nu^*)', vec(\Phi^*)']', \;\; \dim(\Theta_{\mathbb{Q}}) = K\,(1 + Kp)\,,$$

  by minimizing the sum of squared fitting errors between the observed and theoretical yields. In other words, in this second step and for a given $\widehat{\Theta}_{\mathbb{P}}$, we estimate (γ_o, Γ).

- More precisely, in the **scalar case**, we estimate $\Theta_{\mathbb{Q}}$ by nonlinear lest squares (**NLLS**), while, in the **multivariate case**, these parameters are estimated by **Constrained NLLS**.

- The constraints are imposed to satisfy ICCs on (C_h, D_h).

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Estimation Method:

- Given the complete set of 9 maturities of our data base, and given a number m of yields used to estimate the vector of historical parameters $\Theta_{\mathbb{P}}$, we denote by H_m^* the set of remaining maturities used to estimate the vector of risk-neutral parameters $\Theta_{\mathbb{Q}}$.
- In the AR(p) case, w_t is the one-month yield to maturity $R(t, 1m) = r_t$ expressed at a monthly frequency.
- In the bivariate VAR(p) Factor-Based case the factor is given by :

$$w_t = [R(t, 1m), R(t, 60m) - R(t, 1m)]',$$

where $[R(t, 60m) - R(t, 1m)]$ is the spread at date t between the five-year and one-month yield to maturity, expressed at a monthly frequency.

- The NLLS estimator for the AR(p) case, is determined by :

$$\begin{cases} \widehat{\Theta}_{\mathbb{Q}} = Arg \min_{\Theta_{\mathbb{Q}}} S^2(\Theta_{\mathbb{Q}}), \\ S^2(\Theta_{\mathbb{Q}}) = \sum_{t=p}^{T} \sum_{h \in H_1^*} [R^o(t, h) - R(t, h)]^2, \end{cases} \tag{I.21}$$

given the set H_1^* of maturities used to estimate the risk-neutral parameters; $R^o(t, h)$ is the observed yield, while $R(t, h)$ is the model-implied one.

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Estimation Method:

- The constrained NLLS estimator, in our bivariate model specification, is given by :

$$\begin{cases} \widehat{\Theta}_{\mathbb{Q}} = Arg\min_{\Theta_{\mathbb{Q}}} S^2(\Theta_{\mathbb{Q}}) \\ S^2(\Theta_{\mathbb{Q}}) = \sum_{t=p}^{T} \sum_{h \in H_2^*} [R^o(t,h) - R(t,h)]^2, \\ \text{s. t. } \sum_{t=p}^{T} [R^o(t,60m) - R(t,60m)]^2 = 0\,, \end{cases} \tag{I.22}$$

- The constraint in the minimization program (I.22) guarantees the ICCs on the five-year yield to maturity.

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Results for AR(p) Gaussian ATSMs:

- The maximum value of the mean Log-Likelihood and the values of the estimated vector of parameters $\Theta_{\mathbb{P}} = (\nu, \varphi_1, \ldots, \varphi_p, \sigma)'$ of the AR(p) Factor-Based Term Structure models, for $p \in \{1, \ldots, 6\}$, are presented in Tables 2 and 3 [the t-values are given in parenthesis].

- We denote with $mlogL$ the mean log-Likelihood of the AR(p) model: $mlogL = logL(\Theta_{\mathbb{P}}|w_1, \ldots, w_{T-p})/(T-p)$.

- The Akaike Information Criterion (AIC) (for ranking among models) is given by $2mlogL - (2k/(T-p))$, with k denoting the dimension of $\Theta_{\mathbb{P}}$.

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Results for AR(p) Gaussian ATSMs:

Table 2 : AR(p) Factor-Based Term Structure models. Maximum value of the mean Log-Likelihood, AIC and parameter estimates of ν and σ. (**) denotes a parameter significant at 0.05; (*) denotes a parameter significant at 0.1.

	AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)
mlogL	5.9566	5.9587	5.9608	5.9613	5.9722	5.9709
AIC	11.897	11.896	11.895	11.891	11.907	11.899
ν	0.00023	0.00021	0.00023	0.00021	0.00019	0.00019
	[2.672] **	[2.482] **	[2.659] **	[2.476] **	[2.157] **	[2.126] **
$\sigma^2\ 10^4$	0.0039	0.0039	0.0039	0.0039	0.0038	0.0038
	[13.748] **	[13.730] **	[13.712] **	[13.694] **	[13.675] **	[13.657] **

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Results for AR(p) Gaussian ATSMs:

Table 3 : AR(p) Factor-Based Term Structure models Parameter estimates of $(\varphi_1, \ldots, \varphi_p)$. $(^{**})$ denotes a parameter significant at 0.05; $(^{*})$ denotes a parameter significant at 0.1.

	AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)
φ_1	0.9580 ** [65.5620]	0.8798 ** [17.2393]	0.8861 ** [17.1525]	0.8912 ** [17.1688]	0.8814 ** [17.1628]	0.8806 ** [16.9714]
φ_2		0.0811 [1.5938]	0.1547 ** [2.2869]	0.1456 ** [2.0843]	0.1672 ** [2.4260]	0.1675 ** [2.3885]
φ_3			−0.0829 * [−1.6459]	−0.1372 * [−1.9204]	−0.1595 ** [−2.3048]	−0.1586 ** [−2.2623]
φ_4				0.0608 [1.1455]	−0.0790 [−1.1788]	−0.0798 [−1.1240]
φ_5					0.1557 ** [3.1048]	0.1510 ** [2.4443]
φ_6						0.0053 [0.1232]

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Results for AR(p) Gaussian ATSMs:

- An examination of the above displayed parameter estimates show, first of all, that the historical dynamics of the (one-month to maturity) short rate is not Markovian of order one, given that, in the AR(5) and AR(6) specifications, the parameters $(\varphi_1, \varphi_2, \varphi_3, \varphi_5)$ are always significative.

- Table 2 shows also that the constant term ν and of the volatility term σ are significative in each of the examinated models.

- The minimum value of the mean NLLS criterion $[S^2(\widehat{\Theta}_{\mathbb{Q}})/T^*]$ and the values of the estimated vector of risk-neutral parameters $\Theta_{\mathbb{Q}} = (\nu^*, \varphi_1^*, \ldots, \varphi_p^*)$, with $p \in \{1, \ldots, 6\}$, are presented in Tables 5 and 6 [the t-values are given in parenthesis]. We also rank the models in terms of the Root Mean Square Error (RMSE) and Mean Absolute Error (MAE).

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Results for AR(p) Gaussian ATSMs:

Table 4 : AR(p) Factor-Based Term Structure models. Minimum value of the mean NLLS criterion, RMSE, MAE and parameter estimates of ν^*. (**) denotes a parameter significant at 0.05; (*) denotes a parameter significant at 0.1.

	AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)
$\frac{S^2(\widehat{\Theta_{\mathbb{Q}}})}{T^*}\,10^4$	0.0054	0.0051	0.0050	0.0048	0.0047	0.0046
$RMSE\ 10^4$	7.36	7.16	7.09	6.96	6.87	6.79
$MAE\ 10^4$	5.30	5.26	5.28	5.24	5.17	5.09
$\nu^*\ 10^4$	1.10	1.51	1.52	1.48	1.48	1.52
	[33.25] **	[22.60] **	[22.93] **	[22.98] **	[22.71] **	[22.45] **

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Results for AR(p) Gaussian ATSMs:

Table 5 : AR(p) Factor-Based Term Structure models. Parameter estimates of $(\varphi_1^*, \ldots, \varphi_p^*)$. $(^{**})$ denotes a parameter significant at 0.05; $(^{*})$ denotes a parameter significant at 0.1.

	AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)
φ_1^*	0.9899 ** [1877]	0.5076 ** [9.6003]	0.7333 ** [14.2703]	0.7758 ** [15.4922]	0.7382 ** [14.2057]	0.7037 ** [13.3209]
φ_2^*		0.4788 ** [9.1313]	−0.0299 [−0.4132]	0.2291 ** [2.8931]	0.2947 ** [3.6124]	0.2998 ** [3.6802]
φ_3^*			0.2832 ** [7.5221]	−0.3860 ** [−5.3681]	−0.1600 ** [−2.0834]	−0.1069 [−1.3898]
φ_4^*				0.3685 ** [10.2233]	−0.1977 ** [−2.6864]	0.0123 [0.1609]
φ_5^*					0.3126 ** [8.4180]	−0.2173 ** [−2.9386]
φ_6^*						0.2961 ** [7.7697]

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Results for VAR(p) Gaussian ATSMs:

- As in the scalar case, we present the maximum value of the mean Log-Likelihood and the values of the estimated vector of parameters $\Theta_{\mathbb{P}} = [\nu', vec(\Phi)', vech(\Sigma\Sigma')']'$ of the bivariate VAR(p) Factor-Based Term Structure models, for an AR order $p = 1$ and $p = 2$.

- These results are presented in Tables 6 and 7. We have also estimated the historical parameters of the above mentioned bivariate VAR(p) model, for p larger than 2, but the AIC criterion has indicated the first two AR orders as the preferred ones.

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Results for VAR(p) Gaussian ATSMs:

Table 6 : VAR(p) Factor-Based Term Structure models. (**) denotes a parameter significant at 0.05; (*) denotes a parameter significant at 0.1.

	VAR(1)	VAR(2)
mlogL	12.6403	12.6837
AIC	25.2330	25.2984
ν_1	0.000065 [0.5856]	0.000132 [1.2262]
ν_2	0.000080 [0.8157]	0.000026 [0.2701]
σ_1^2	0.00000039 [5.94750] **	0.00000036 [6.02614] **
σ_{21}	-0.00000028 [-6.0995] **	-0.00000026 [-6.2100] **
σ_2^2	0.00000030 [7.6713] **	0.00000028 [8.0731] **

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Results for VAR(p) Gaussian ATSMs:

Table 7 : VAR(1) and VAR(2) Factor-Based Term Structure models. Parameter estimates of (φ_1, φ_2). (**) denotes a parameter significant at 0.05; (*) denotes a parameter significant at 0.1.

	VAR(1)		VAR(2)	
Φ_1	0.9742	0.0719	1.3318	0.6207
	[59.8835] **	[2.2174] **	[15.0111] **	[7.0095] **
	0.0091	0.8769	−0.2744	0.4353
	[0.6388]	[30.7835] **	[-3.4988] **	[5.5601] **
Φ_2			−0.3648	−0.5762
			[-3.6117] **	[-5.8201] **
			0.2893	0.4642
			[3.2397] **	[5.3020] **

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Results for VAR(p) Gaussian ATSMs:

- If we consider the parameter estimates of Tables 6 and 7, we observe that the joint historical dynamics of short rate and spread is not Markovian of order one, given that, in the VAR(2) specification, the parameters in the second autoregressive matrix φ_2 are significantly different from zero.

- Moreover, the AIC indicates this model as the preferred one. Table 6 shows also that the constant term $(\nu_1, \nu_2)'$ is not significative for both AR orders.

- We present the minimum value of the mean NLLS criterion $[S^2(\widehat{\Theta}_{\mathbb{Q}})/T^*]$ and the values of the estimated vector of risk-neutral parameters $\Theta_{\mathbb{Q}} = [(\nu^*)', vec(\Phi^*)']'$, for the bivariate VAR(1) and VAR(2) Factor-Based Term Structure models, in Tables 8 and 9.

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Results for VAR(p) Gaussian ATSMs:

Table 8 : VAR(p) Factor-Based Term Structure models. Minimum value of the mean NLLS criterion, RMSE, MAE and parameter estimates of (ν_1^*, ν_2^*). $(^{**})$ denotes a parameter significant at 0.05; $(^{*})$ denotes a parameter significant at 0.1.

	VAR(1)	VAR(2)
$S^2(\hat{\theta}_{\mathbb{Q}})/T^*$	0.00000009	0.00000008
RMSE	0.000297	0.000283
MAE	0.000208	0.000198
ν_1^*	-0.000058	-0.000055
	$[-6.6459]$ **	$[-4.9423]$ **
ν_2^*	0.000072	0.000071
	[5.7860] **	[4.5783] **

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Results for VAR(p) Gaussian ATSMs:

Table 9 : VAR(1) and VAR(2) Factor-Based Term Structure models. Parameter estimates of (Φ_1^*, Φ_2^*). $(^{**})$ denotes a parameter significant at 0.05; $(^{*})$ denotes a parameter significant at 0.1.

	VAR(1)		VAR(2)	
Φ_1^*	1.0131	0.1105	1.3154	0.6020
	[805.8869] **	[34.5743] **	[28.4716] **	[9.5120] **
	−0.0156	0.9072	−0.2528	0.4142
	[-8.6611] **	[203.2978] **	[-3.5778] **	[4.2509] **
Φ_2^*			−0.3004	−0.4890
			[-6.5177] **	[-7.8923] **
			0.2342	0.4839
			[3.3244] **	[5.0769] **

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

Results for VAR(p) Gaussian ATSMs:

- We find that, also in this bivariate risk-neutral (pricing) framework, the lagged values of the short rate and spread play an important role in the model specification. One may observe the significativity of all risk-neutral AR coefficients in the VAR(2) specification.
- In other words, a VAR(2) specification for the historical and risk-neutral dynamics of the factor driving term structure shapes, lead to propose a bivariate term structure model which is able to fit yields to maturity better than the VAR(1) and AR(p) specification.
- Summary of in-sample fit (using *RMSE* and *MAE*). The bivariate setting strongly dominate the scalar one, regardless the number of lags.
- In the bivariate setting, the introduction of an additional lag (marginally) improves the fitting performance.

	AR(6)	VAR(1)	VAR(2)
RMSE	0.000679	0.000297	0.000283
MAE	0.000509	0.000208	0.000198

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

- This is the most popular simple model of the term structure, and we distinguish between the *pure expectation hypothesis* (PEH) and the *expectation hypothesis* (EH).
- The PEH says that the expected excess returns on long-term over short-term bonds are zero. The EH allows expected excess returns on the long-term bond to depend on the maturity but not on time.

a) The one-period continuously compounded yield $R(t,1)$ (which is the same as the one-period geometric return on a one-period bond) should equal the expected geometric return on a $(T-t)$-period bond held for one period:

$$R(t,1) = \mathbb{E}_t[\ln(B(t+1, T-t-1)/B(t, T-t))] = \mathbb{E}_t[r_{t,t+1}^{(T-t)}]$$

b) A long-term $(T-t)$-period yield $R(t, T-t)$ should equal the expected sum of $(T-t)$ successive one-period yields (i.e. returns) rolled over for $(T-t)$ periods:

$$R(t, T-t) = \frac{1}{T-t} \sum_{i=0}^{T-t-1} \mathbb{E}_t[R(t+i,1)]$$

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

- The EH theory says that (using the cont comp notation) this difference:

$$R(t, T-t) - \frac{1}{T-t} \sum_{i=0}^{T-t-1} \mathbb{E}_t[R(t+i, 1)] =: TP(t, T),$$

  called *Term Premia* of the yield $R(t, T-t)$, is constant over time and depends only on $T-t$: $TP(t, T-t) = TP(T-t)$. The PEH says that $TP(t, T) = 0$.
- $\frac{1}{T-t} \sum_{i=0}^{T-t-1} \mathbb{E}_t[R(t+i, i+1)]$ is called *Expectation Component* of $R(t, T-t)$ and it is denoted $EX(t, T)$. This part of the yield provides information about **market expectations (forecast !) on future short rates** over an horizon given by the residual maturity $T-t$.
- Thus, $TP(t, T)$ is the amount by which the long rate $(R(t, T))$ exceeds the expected return from investing in a series of short-term instruments.
- It is a **risk compensation** required to hold a long-term instead of a short-term debt.

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

- What does the empirical evidence suggest about the Expectation Hypothesis ?
- Given the one-period geometric bond return $r_{t,t+1}^{(h)} = \ln \dfrac{B(t+1, T-t-1)}{B(t, T-t)}$, the one-period excess bond return is $ex_{t+1}^{(h)} = r_{t,t+1}^{(h)} - R(t,1)$ (with $h = T - t$). We can equivalently write:

$$ex_{t+1}^{(h)} = -[R(t+1, T-t-1) - R(t, T-t)]\,(T-t-1) + [R(t, T-t) - R(t,1)]$$

and thus:

$$R(t+1, T-t-1) - R(t, T-t) = -\frac{1}{T-t-1}\, ex_{t+1}^{(h)} + \frac{1}{T-t-1}\,[R(t, T-t) - R(t,1)]$$

- The EH states $\mathbb{E}_t(ex_{t+1}^{(h)}) = 0$, and to test for it we can run the following regression:

$$\begin{aligned} R(t+1, T-t-1) - R(t, T-t) &= \phi_0(h,1) \\ &+ \phi_1(h,1)\,\frac{1}{T-t-1}\,[R(t, T-t) - R(t,1)] + u_{t+1}(h,1) \end{aligned}$$

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

- It is also possible to prove that, given an m-period bond return ($m < T - t$)
  $r^{(h)}_{t,t+m} = \frac{1}{m} \ln \frac{B(t+m, T-t-m)}{B(t, T-t)}$, with $ex^{(h)}_{t+m} = r^{(h)}_{t,t+m} - R(t,m)$:

$$R(t+m, T-t-m) - R(t, T-t) = \phi_0(h,m) + \phi_1(h,m) \frac{m}{T-t-m} [R(t, T-t) - R(t,m)] + u^{(h,m)}_{t+m}$$

  These regressions are called in the literature **Campbell-Shiller regressions**, from the paper by Campbell and Shiller (1991, RES).
- If we focus on the first regression ($m = 1$), testing the EH means testing that $\phi_0(h,1) = 0$ and $\phi_1(h,1) = 1$. A well known empirical feature of the U.S. yield curves is that $\widehat{\phi_1}(h,1) < 0$, for all residual maturities and (in general) $|\widehat{\phi_1}(h,1)|$ increasing as far as h increases.
- This is called the **violation of the EH Theory** or **EH Puzzle**.
- Let us see some empirical result about $\widehat{\phi_1}(h,1)$. We take the GSW (2007) data base with quarterly observations from $1964 : Q1$ to $2007 : Q2$.

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

h	3-Q	4-Q	8-Q	12-Q	20-Q	40-Q
$\widehat{\phi_1}(h,1)$	-0.49	-0.74	-0.98	-1.20	-1.55	-2.57
st. dev.	[0.28]	[0.40]	[0.68]	[0.82]	[0.93]	[1.19]

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

- Let us see an application to CRSP data set on the U. S. term structure of interest rates. The sample covers the period from June 1964 to December 1995. We have 379 monthly observations for each of the nine maturities : 1, 3, 6 and 9 months and 1, 2, 3, 4 and 5 years.
- How do $\widehat{\phi_1}(h, m)$ behaves ?

Short Horizon	$m = 3$ months	$m = 6$ months	$m = 9$ months
$h = 6$ months	-0.6942 (0.2533)		
$h = 9$ months	-0.8863 (0.3238)	-0.4023 (0.2429)	
$h = 12$ months	-1.3226 (0.3530)	-0.7867 (0.2381)	-0.4371 (0.1312)
Long Horizon	$m = 1$ year	$m = 2$ years	$m = 3$ years
$h = 4$ years	-1.8078 (0.2981)	-0.8380 (0.2889)	-0.0421 (0.2682)
$h = 5$ years	-1.7470 (0.3291)	-0.9720 (0.3199)	-0.2378 (0.3283)

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

Short Horizon Expectation Hypothesis Tests : lags are useful !

Short Horizon	$m = 3$ months	$m = 6$ months	$m = 9$ months
$h = 6$ months	-0.6942 (0.2533)		
2-Factor VAR(1)	0.5828 (0.3485)		
2-Factor VAR(2)	-0.3800 (0.3837)		
$h = 9$ months	-0.8863 (0.3238)	-0.4023 (0.2429)	
2-Factor VAR(1)	0.4133 (0.3469)	0.4722 (0.2693)	
2-Factor VAR(2)	-0.5480 (0.3960)	-0.3890 (0.3182)	
$h = 12$ months	-1.3226 (0.3530)	-0.7867 (0.2381)	-0.4371 (0.1312)
2-Factor VAR(1)	0.2454 (0.3486)	0.3187 (0.2710)	0.3796 (0.2430)
2-Factor VAR(2)	-0.6935 (0.4069)	-0.5272 (0.3248)	-0.3675 (0.2930)

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

Long Horizon Expectation Hypothesis Tests: some problem !

Long Horizon	$m = 1$ year	$m = 2$ years	$m = 3$ years
$h = 4$ years	-1.8078 (0.2981)	-0.8380 (0.2889)	-0.0421 (0.2682)
2-Factor VAR(1)	-0.8569 (0.3536)	-0.0085 (0.3414)	0.8626 (0.3514)
2-Factor VAR(2)	-1.4088 (0.4084)	-0.2338 (0.3864)	0.9368 (0.3843)
$h = 5$ years	-1.7470 (0.3291)	-0.9720 (0.3199)	-0.2378 (0.3283)
2-Factor VAR(1)	-1.1444 (0.4102)	-0.0033 (0.3953)	1.1279 (0.3970)
2-Factor VAR(2)	-1.6686 (0.4635)	-0.2112 (0.4373)	1.2060 (0.4267)

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I.7 – EMPIRICAL ANALYSIS OF GAUSSIAN ATSMs

A benchmark: the Expectation Hypothesis Theory

- Which "directions" should we follow to improve the empirical performances of a given model ?

i) Adding **new factors** (latent and/or observable) or **sources of non-linearities** (stochastic volatility, jumps, switching of regimes) able to explain the strong persistence in yields [see Dai, Singleton and Yang (2007, RFS), Monfort and Pegoraro (2007, JFEC), Monfort and Renne (2012, JFEC) and Gourieroux, Monfort, Pegoraro and Renne (2013, JFEC)].

ii) Estimating model parameters in a way coherent with interest rates **persistence** [see Jardet, Monfort and Pegoraro (2012, JBF)].

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I.8 – AN ALTERNATIVE ESTIMATION PROCEDURE

Adrian, Crump and Moench (2013, JFE)

- As we have previously seen, a Gaussian ATSM with VAR(1) factor w_t:

$$w_{t+1} = \nu + \Phi w_t + \Sigma \varepsilon_{t+1} ,$$

and with exponential-affine SDF ($\Gamma_t = \gamma_o + \gamma\, w_t$):

$$M_{t,t+1} = \exp\left[-r_t + \Gamma_t' \varepsilon_{t+1} - \frac{1}{2}\Gamma_t'\Gamma_t\right] ,$$

has a one-period geometric bond return following:

$$\begin{aligned} \rho(t+1,\ T) \quad &= \quad r_t - \tfrac{1}{2}\,\omega(t+1,\ T)'\omega(t+1,\ T) + \omega(t+1,\ T)'\Gamma_t \\ &\quad -\omega(t+1,\ T)'\,\varepsilon_{t+1} , \end{aligned}$$

where $\omega(t+1,\ T) = -(\Sigma' C_{T-t-1})$ is an K-dimensional vector.

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I.8 – AN ALTERNATIVE ESTIMATION PROCEDURE

Adrian, Crump and Moench (2013, JFE)

- Adrian, Crump and Moench (2013) exploit the fact that the one-period excess bond return:

$$rx_{t+1}^{(h-1)} := \log B(t+1, t+h) - \log B(t, t+h) - r_t$$

is conditionally Gaussian and linear in (γ_o, γ), in order to make their estimation computationally fast, even for a large number of factors.

- From $B(t,h) = \mathbb{E}_t[M_{t,t+1}\, B(t+1, h-1)]$ we find:

$$1 = \mathbb{E}_t\left[\exp\left(rx_{t+1}^{(h-1)} - \frac{1}{2}\Gamma_t'\Gamma_t - \Gamma_t'\varepsilon_{t+1}\right)\right]$$

- Assuming $\{rx_{t+1}^{(h-1)}, \varepsilon_{t+1}\}$ are jointly Gaussian:

$$\begin{aligned}
\mathbb{E}_t\left[rx_{t+1}^{(h-1)}\right] &= \mathbb{C}ov_t\left(rx_{t+1}^{(h-1)}, \varepsilon_{t+1}'\Gamma_t\right) - \frac{1}{2}\mathbb{V}ar_t\left(rx_{t+1}^{(h-1)}\right) \\
&= \mathbb{C}ov_t\left(rx_{t+1}^{(h-1)}, \varepsilon_{t+1}'\right)(\gamma_o + \gamma\, w_t) - \frac{1}{2}\mathbb{V}ar_t\left(rx_{t+1}^{(h-1)}\right) \\
&= \beta_t^{(h-1)'}(\gamma_o + \gamma\, w_t) - \frac{1}{2}\mathbb{V}ar_t\left(rx_{t+1}^{(h-1)}\right),
\end{aligned}$$

where $\beta_t^{(h-1)'} := \mathbb{C}ov_t\left(rx_{t+1}^{(h-1)}, \varepsilon_{t+1}'\right)$.

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I.8 – AN ALTERNATIVE ESTIMATION PROCEDURE

Adrian, Crump and Moench (2013, JFE)

- Now, we can always decompose $rx_{t+1}^{(h-1)}$ into an expected component and an unexpected one:

$$rx_{t+1}^{(h-1)} = \mathbb{E}_t\left[rx_{t+1}^{(h-1)}\right] + \left(rx_{t+1}^{(h-1)} - \mathbb{E}_t\left[rx_{t+1}^{(h-1)}\right]\right)$$

and the latter can be further decomposed into a component cond. correlated with ε_{t+1} and another that is cond. orthogonal to ε_{t+1}:

$$rx_{t+1}^{(h-1)} - \mathbb{E}_t\left[rx_{t+1}^{(h-1)}\right] = \beta_t^{(h-1)'}\,\varepsilon_{t+1} + e_{t+1}^{(h-1)}$$

where the return pricing errors $e_{t+1}^{(h-1)}$ are cond. i.i.d. with variance σ^2.

- Given that $\mathbb{V}ar_t\left(rx_{t+1}^{(h-1)}\right) = \beta_t^{(h-1)'}\beta_t^{(h-1)} + \sigma^2$, we can thus write:

$$rx_{t+1}^{(h-1)} = \beta_t^{(h-1)'}\left(\gamma_o + \gamma\, w_t\right) - \frac{1}{2}\left(\beta_t^{(h-1)'}\beta_t^{(h-1)} + \sigma^2\right) + \beta_t^{(h-1)'}\varepsilon_{t+1} + e_{t+1}^{(h-1)}\,.$$

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I.8 – AN ALTERNATIVE ESTIMATION PROCEDURE

Adrian, Crump and Moench (2013, JFE)

- In their baseline model they assume w_t is observable and made of a linear combination of yields, such as Principal Components (PCs) (see Section I.9). They estimate model parameters using holding period returns based on the same set of yields.
- Stacking the system across $h \in \{2, \ldots, H\}$ maturities and $t \in \{1, \ldots, T-1\}$ time periods, we rewrite it as:

$$rx = \beta' \left(\gamma_o \mathbf{1}'_{T-1} + \gamma X_-\right) - \frac{1}{2}\left(B^* vec(I_K) + \sigma^2 \mathbf{1}_{H-1}\right)\mathbf{1}'_{T-1} + \beta'\mathcal{E} + E$$

where:

- rx is a $(H-1, T-1)$ matrix of excess returns;
- $\beta = \left[\beta^{(2)} \mid \beta^{(3)} \mid \ldots \mid \beta^{(H)}\right]$ is a $(K, H-1)$ matrix of factor loadings;
- $\mathbf{1}_\ell$ is an ℓ-dimensional vector of ones;
- $X_- = \left[w_1, \ldots, w_{T-1}\right]$ is a $(K, T-1)$ matrix of lagged pricing factors;
- $B^* = \left[vec(\beta^{(2)}\beta^{(2)'}) \mid vec(\beta^{(3)}\beta^{(3)'}) \mid \ldots \mid vec(\beta^{(H)}\beta^{(H')})\right]'$ is a $(H-1, K^2)$ matrix;
- $\mathcal{E}$ is a $(K, T-1)$ matrix of normalized residuals $\{\widehat{\varepsilon_t}\}$, and E is a $(H-1, T-1)$ matrix of $\{\widehat{e_t^{(h)}}\}$ residuals.

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I.8 – AN ALTERNATIVE ESTIMATION PROCEDURE

Adrian, Crump and Moench (2013, JFE)

- Estimation Procedure:

1st) We estimate $\theta_{\mathbb{P}} = (\nu, \Phi, \Sigma)$ by OLS, and then we build $\mathcal{E}$ from $\{\widehat{\varepsilon_t}\}$

2nd) Run the following regression:

$$\begin{aligned} rx &= \mathbf{a}\,\mathbf{1}'_{T-1} + \beta'\mathcal{E} + \mathbf{c}X_- + E \\ \mathbf{a} &= \beta'\gamma_o - \frac{1}{2}\left(B^* vec(I_K) + \sigma^2\,\mathbf{1}_{H-1}\right) \\ \mathbf{c} &= \beta'\gamma \end{aligned}$$

in order to obtain:

$$\left[\widehat{\mathbf{a}}\,|\,\widehat{\beta}'\,|\,\widehat{\mathbf{c}}\right] = rx\,\widetilde{Z}'(\widetilde{Z}\,\widetilde{Z}')^{-1} \quad \texttt{where} \quad \widetilde{Z} = \left[\mathbf{1}_{T-1}\,|\,\widehat{\mathcal{E}}'\,|\,X'_-\right]',$$

we collect the associated residuals in $\widehat{E}$, in order to calculate $\widehat{\sigma}^2 = Tr(\widehat{E}\,\widehat{E}')/(H-1)(T-1)$, and we calculate $\widehat{B}^*$ from $\widehat{\beta}'$.

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I.8 – AN ALTERNATIVE ESTIMATION PROCEDURE

Adrian, Crump and Moench (2013, JFE)

3rd) Given the previously estimated parameters, we easily find:

$$\widehat{\gamma} = \left(\widehat{\beta}\widehat{\beta}'\right)^{-1} \widehat{\beta}\widehat{\mathbf{c}}$$

and from $\widehat{\beta}'\gamma_o = \widehat{\mathbf{a}} + \frac{1}{2}\left(\widehat{B}^* vec(I_K) + \widehat{\sigma}^2 \mathbf{1}_{H-1}\right)$ we retrieve:

$$\widehat{\gamma}_o = \left(\widehat{\beta}\widehat{\beta}'\right)^{-1} \widehat{\beta}\left[\widehat{\mathbf{a}} + \frac{1}{2}\left(\widehat{B}^* vec(I_K) + \widehat{\sigma}^2 \mathbf{1}_{H-1}\right)\right].$$

- From the estimated model parameters, we can generate e zero-coupon yield curve using the recursive equations (C_h, D_h) and an identification between $\beta^{(h)}$ and C_h is easily found.

- Joslin, Singleton and Zhu (2011, RFS) propose an alternative computationally fast MLE of Gaussian ATSMs. See also Joslin, Le and Singleton (2013, JFE) and Joslin, Priebsch and Singleton (2014, JF).
- In the next slides we quickly present some paper in the literature where other estimation techniques are adopted. Those techniques will be presented in one of the next lectures.

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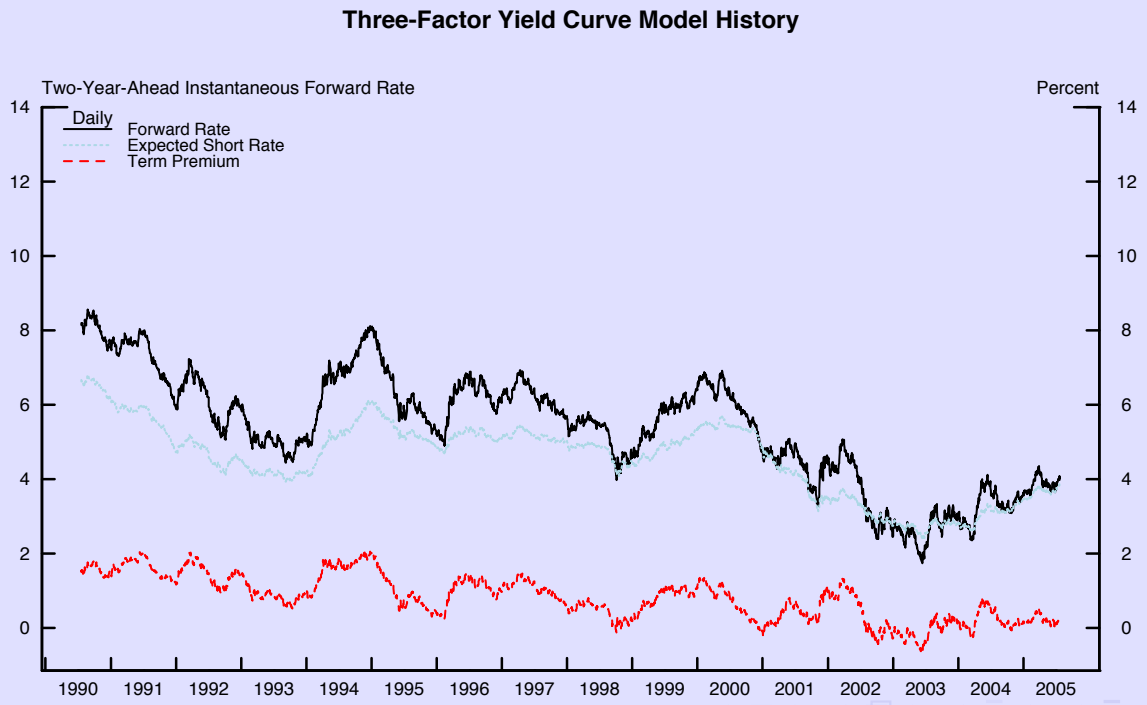
Example I.2 (Kim and Wright (2005))

- *Three-factor "yield-only model" (no macro variables, except inflation in one variant of the model).*
- *Short-term rate:* $r_t = \omega_0 + \omega_{1,1} w_{1,t} + \omega_{1,2} w_{2,t} + \omega_{1,3} w_{3,t}$.
- *Outputs are regularly updated by the Federal Reserve Board.*
- *Model estimated by Kalman filter; the state-space model includes survey-based variables.*

  Monthly data on the 6-month and 12-month-ahead forecasts of the three-month T-Bill yield from Blue Chip Financial Forecasts and semiannual data on the average expected three-month T-Bill yield from 6 to 11 years.



Example I.3 (Ang and Piazzesi (2003))

- *Ang and Piazzesi (2003): one of the first paper mixing latent and macrovariables.*
- *The set up is also of the form Gaussian VAR, except that the VAR features several lags.*
- $w_t = [f^o_{1,t}, f^o_{2,t}, f^u_{1,t}, f^u_{2,t}, f^u_{3,t}]'$ *where:*
  - $f^o_{1,t}$ *is the first Principal Component of a set of 3 price indexes (growth rates)*
  - $f^o_{2,t}$ *is the first Principal Component of a set of 4 real activity proxies (HELP, EMPLOY, IP, UE).*
  - $f^u_{i,t}$ *are unobserved, or latent, factors.*
- *Nominal short-term rate follows a Taylor rule.*
- *Latent factors are estimated via "inversion techniques".*

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Example I.4 (Joslin, Priebsch and Singleton (2014))

- *Models stating that r_t affine in macro-factors imply that macro-factors are "spanned" by the yield curve: macro-factors should be perfectly explained by yields of different maturities. Not the case in the data.*
- *Joslin Priebsch and Singleton (2014): "unspanned macro risks".*
- $w_t = [\mathcal{P}_t', \mathcal{M}_t']'$, *where $\mathcal{P}_t$ are yield factors and $\mathcal{M}_t$ are macro factors.*
- *Model:*

$$
\begin{aligned}
r_t &= \omega_0 + \omega_{\mathcal{P}}' \mathcal{P}_t \\
\begin{bmatrix} \mathcal{P}_t \\ \mathcal{M}_t \end{bmatrix} &= \begin{bmatrix} \Phi_{\mathcal{P}\mathcal{P}} & \Phi_{\mathcal{P}\mathcal{M}} \\ \Phi_{\mathcal{M}\mathcal{P}} & \Phi_{\mathcal{M}\mathcal{M}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{t-1} \\ \mathcal{M}_{t-1} \end{bmatrix} + \Sigma \varepsilon_t \\
\begin{bmatrix} \mathcal{P}_t \\ \mathcal{M}_t \end{bmatrix} &= \mu + \begin{bmatrix} \Phi^{\mathbb{Q}}_{\mathcal{P}\mathcal{P}} & 0 \\ \Phi^{\mathbb{Q}}_{\mathcal{M}\mathcal{P}} & \Phi^{\mathbb{Q}}_{\mathcal{M}\mathcal{M}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{t-1} \\ \mathcal{M}_{t-1} \end{bmatrix} + \Sigma \varepsilon_t^{\mathbb{Q}},
\end{aligned}
$$

  where ε_t and $\varepsilon_t^{\mathbb{Q}}$ are $\mathcal{N}(0, Id)$ under $\mathbb{P}$ and $\mathbb{Q}$, respectively.
- *$\mathcal{M}_t$ does not Granger-cause $\mathcal{P}_t$ under $\mathbb{Q}$ and r_t is affine in $\mathcal{P}_t$ (only) $\Rightarrow$ yields $R(t,h)$ are affine in $\mathcal{P}_t$ (only).*
- *However $\mathcal{M}_t$ does Granger-cause $\mathcal{P}_t$ under $\mathbb{P}$ $\Rightarrow$ macro-shocks affect $\mathcal{P}_t$ and thus the future yield curve.*

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Example I.5 (Ang, Boivin, Dong and Loo-Kung (2011))

- *Ang, Boivin, Dong and Loo-Kung (2011): macro-finance model based on a quadratic framework.*
- *Taylor rule with time-varying parameters:*

$$r_t = \omega_0 + a_t g_t + b_t \pi_t,$$

where $x_t = (g_t, \pi_t, a_t, b_t)'$ *follows a Gaussian VAR.*

- *This is the context described in Example I.8 of Lecture 1 (Quadratic Gaussian process). We will look at Quadratic term structure models in Lecture 4.*
- r_t *is linear in* $w_t = (x_t, vec(x_t x_t')')'$. *Specifically:*

$$r_t = \omega_0 + \omega_1' w_t,$$

with $\omega_1 = [v, vec(V)]'$, *where*

$$v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}.$$

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I.9 – PCA OF THE YIELD CURVE

- The factors we used in the previous sections were given yields (short rate, long rate, spreads) or Principal Components (PCs) like in ACM (2013). But, what PCs are ? They are factors :

1. able to explain a very large proportion of the variance of the yields across the maturity spectrum,
2. and independent of each other.

- **Principal Component Analysis (PCA)** is a statistical technique that identifies the best factors from historical yields, where the term 'best' is in the sense of the two conditions mentioned above.

- It is used to extract the factors explaining most of the variability of the multivariate phenomenon we observe : a n-dimensional random process $X_t = (X_{1,t}, \ldots, X_{n,t})'$ observed over time.

- PCA produces a lower dimensional description of $(X_t)_{t=1,2,\ldots}$ searching for linear combinations of X_t with the largest variances [see Hardle and Simar (2003) : *Applied Multivariate Statistical Analysis*, Springer].

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I.9 – PCA OF THE YIELD CURVE

- The key mathematical principles behind the PCA are the **Spectral Decomposition theorems** of linear algebra.

- **Eigenvalue-Eigenvector Decomposition:** *Each symmetric* $(n \times n)$ *matrix* $\mathcal{A}$ *can be written as:*

$$\mathcal{A} = \Gamma\Lambda\Gamma' = \sum_{j=1}^{n} \lambda_j \gamma_j \gamma_j' \,, \text{ where } \Lambda = diag(\lambda_1, \ldots, \lambda_n) \text{ contains the}$$

n eigenvalues of the matrix $\mathcal{A}$*, and where* $\Gamma = (\gamma_1, \ldots, \gamma_n)$

is an orthogonal matrix ($\Gamma^{-1} = \Gamma'$*, i.e.* $\gamma_i'\gamma_j = 0 \; \forall i \neq j$

with $||\gamma_j|| = 1$) *where the* j^{th} *column is the* j^{th} *eigenvector* γ_j *of* $\mathcal{A}$

Useful application : if $\mathcal{A} = \Gamma\Lambda\Gamma'$ then, for any $\alpha \in \mathbb{R}$, we have $\mathcal{A}^\alpha = \Gamma\Lambda^\alpha\Gamma'$ where $\Lambda^\alpha = diag(\lambda_1^\alpha, \ldots, \lambda_n^\alpha)$.

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I.9 – PCA OF THE YIELD CURVE

- **Singular Value Decomposition Theorem (SVD) :** Each $(k \times n)$ matrix $\mathcal{A}$, with $rank(\mathcal{A}) = r$, can be decomposed as:

  $\mathcal{A} = \Gamma \Lambda \Delta'$, where Γ is $(k \times r)$ and Δ is $(n \times r)$.

  Both Γ and Δ are column orthonormal, i.e. $\Gamma'\Gamma = \Delta'\Delta = I_r$.

  $\Lambda = diag(\lambda_1^{1/2}, \ldots, \lambda_r^{1/2})$, $\lambda_j > 0$. The values $\lambda_1, \ldots, \lambda_r$ are the non-zero eigenvalues of the matrices $\mathcal{A}\mathcal{A}'$ and $\mathcal{A}'\mathcal{A}$.

  Γ and Δ consist of the corresponding r eigenvectors of these matrices.

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I.9 – PCA OF THE YIELD CURVE

- **Principal Components of a Random Vector :** let us consider a $\mathbb{R}^n$-valued square-integrable random vector $X = (x_1, \ldots, x_n)'$ with mean vector $\mu = \mathbb{E}[X]$ and variance-covariance matrix $\Sigma = \mathbb{V}[X]$.

- Since Σ is symmetric and positive semi-definite, then from Eigenvalue-Eigenvector Decomposition we have that $\Sigma = \Gamma\Lambda\Gamma'$ with $\lambda_i \geq 0 \; \forall i \in \{1, \ldots, n\}$.

- The **principal components transform** of X is defined as:

  $Y = \Gamma'(X - \mu)$, which can be seen as a re-centering and rotation of X, and each $Y_i = \gamma_i'(X - \mu)$ is the projection of $X - \mu$ onto the ith eigenvector γ_i of Γ. $Y_i = i^{th}$ *principal component of* X. γ_i is also called the ith vector of *loadings* of X.

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I.9 – PCA OF THE YIELD CURVE

- We thus obtain the decomposition:

$$X = \mu + \Gamma Y = \mu + \sum_{j=1}^{n} \gamma_j Y_j$$

and we have that $\mathbb{E}[Y] = 0$ and $\mathbb{V}[Y] = \Gamma'\Sigma\Gamma = \Gamma'(\Gamma\Lambda\Gamma')\Gamma = \Lambda$.

- This means that the principal components of X are uncorrelated and have variances $\mathbb{V}[Y_j] = \lambda_j \ \forall j$, which can be ordered from the largest to the smallest:

$$\mathbb{V}[Y_1] = \lambda_1 \geq \ldots \geq \mathbb{V}[Y_n] = \lambda_n \geq 0\,.$$

- It can be shown (exercise!) that the 1st principal component, Y_1, has maximal variance among all standardized linear combinations of X. That is:

$$\mathbb{V}[\gamma_1' X] = \max_{\{\delta : ||\delta|| = 1\}} \{\mathbb{V}[\delta' X]\}$$

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I.9 – PCA OF THE YIELD CURVE

- For $j \in \{2, \ldots, n\}$, the jth principal component Y_j can be shown to have maximal variance among all such linear combinations that are orthogonal to the first $(j-1)$ linear combinations.

- We also have that:

$$\sum_{j=1}^{n} \mathbb{V}[X_j] = trace(\Sigma) = \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \mathbb{V}[Y_j]$$

- The quantity:

$$\sum_{j=1}^{k} \lambda_j / \sum_{j=1}^{n} \lambda_j$$

represents the amount of variability in X explained by the first k principal components $Y_1, \ldots, Y_k$.

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I.9 – PCA OF THE YIELD CURVE

- **Sample Principal Components :** Now let assume to observe our n-dimensional vector over time. Let us therefore denote with $x_t = (x_{1,t}, \ldots, x_{n,t})'$ the date t observation (realization) of the random vector $X_t = (X_{1,t}, \ldots, X_{n,t})'$, and let us assume to have T observations.

- We organize the T observations of each component $x_{i,t}$ in the following $(T \times n)$ matrix: $\mathbf{X} = [x_1', \ldots, x_T']$ where each rows $x_j' = (x_{1,j}, \ldots, x_{n,j})'$ is a sample realization (observation) of the random vector X_t at date $t = j$.

- Let us assume that X_t is independently and identically distributed (i.i.d.) with $\mathbb{E}[X_t] = \mu$ and $\mathbb{V}[X_t] = \Sigma \ \forall t$.

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I.9 – PCA OF THE YIELD CURVE

- We consider the empirical $(T \times T)$ covariance matrix:

$$\widehat{\Sigma}_{ij} = \widehat{\mathrm{Cov}}[X_{i,t}, X_{j,t}] = \frac{1}{T}\sum_{t=1}^{T}(x_{i,t} - \widehat{\mu}_i)(x_{j,t} - \widehat{\mu}_j)$$

where $\widehat{\mu} = (\widehat{\mu}_1, \ldots, \widehat{\mu}_n)' = \frac{1}{T}\sum_{t=1}^{T} x_t$ denotes the empirical mean.

- Given that $\widehat{\Sigma}$ is positive semi-definite, the above PCA applies :

$$\widehat{\Sigma} = \widehat{\Gamma}\widehat{\Lambda}\widehat{\Gamma}' = \sum_{j=1}^{n} \widehat{\lambda}_j \widehat{\gamma}_j \widehat{\gamma}_j' \,, \text{ where } \widehat{\Lambda} = diag(\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n) \text{ and where}$$

$\widehat{\Gamma} = (\widehat{\gamma}_1, \ldots, \widehat{\gamma}_n)$ is an orthogonal matrix

$(\widehat{\Gamma}^{-1} = \widehat{\Gamma}'$, i.e. $\widehat{\gamma}_i'\widehat{\gamma}_j = 0 \ \forall i \neq j)$

with j^{th} column the jth eigenvector $\widehat{\gamma}_j$ of $\widehat{\Sigma}$.

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I.9 – PCA OF THE YIELD CURVE

- The first empirical principal component is the T-dimensional vector $\mathbf{y_1}$ given by:

$$\mathbf{y_1} = (\mathbf{X} - 1_T \widehat{\mu}')\widehat{\gamma}_1 \,.$$

- We thus can write the n empirical principal components in the following compact form:

$$\mathbf{Y} = (\mathbf{X} - 1_T \widehat{\mu}')\widehat{\Gamma}\,, \quad \text{where } \mathbf{Y} = (\mathbf{y_1}, \ldots, \mathbf{y_n}) \text{ is a } (T \times n) \text{ matrix}\,.$$

$$\mathrm{Cov}[y_i, y_j] = \frac{1}{T}\sum_{t=1}^{T} y_{i,t} y_{j,t} = \widehat{\lambda}_i \text{ iif } i = j\,, (= 0 \text{ otherwise})\,.$$

- Remember that the PCA is sensitive to scale changes: if we multiply one variable by a scalar we obtain different eigenvalues and eigenvectors.

- Thus the PCA should be applied to data having the same scale in each variable. Otherwise, we have to use the Normalized Principal Component Analysis (NPCA).

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I.9 – PCA OF THE YIELD CURVE

- Now, let us assume that our $(T \times n)$ data matrix $\mathbf{X} = [x_1', \ldots, x_T']$ contains at each row $x_t' = (x_{1,t}, \ldots, x_{n,t})'$ the sample realization (observation) of the following random vector containing, at date t, the increments of spot rate curve that is:

$$x_{i,t} = r(t, t + h_i)$$

$$\text{with } \; h_i \in \{h_1, \ldots, h_n\} \; \text{ set of maturities}.$$

- The first difference is also (frequently) taken (as a standard practice) given that interest rates are not i.i.d..

- They are highly serially dependent, i.e. $\mathrm{Cov}[r(t, t + h_i), r(t + k, t + h_i)] >> 0$ for $k \neq 0$ (and quite close to one!). Nevertheless, results do not change a lot.

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I.9 – PCA OF THE YIELD CURVE

- Now, let us apply the PCA to the US yields, observed quarterly from 1964:Q1 to 2009:Q4 [from the Gurkaynak, Sack and Write data base], with maturities 3 months, 6 months, 9 months, 1 year, 5 years and 10 years. We find:

PC	Explained Variance (%)	Cumulative Variance (%)	Eigenvalue
1	0.9592	0.9592	5.7554
2	0.0376	0.9968	0.2255
3	0.0025	0.9993	0.0148
4	0.0006	0.9999	0.0038
5	0.0001	1.0000	0.0005
6	0.0000	1.0000	5.02E-06

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I.9 – PCA OF THE YIELD CURVE

- The first 3 eigenvector associated to the first 3 PC are:

$r(t, t+\tau_i)$	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$
3m	0.4067	-0.4178	0.7569
6m	0.4124	-0.3011	-0.1058
9m	0.4141	-0.2207	-0.3442
1y	0.4151	-0.1505	-0.4350
5y	0.4074	0.4341	-0.1419
10y	0.3934	0.6892	0.2968

- The last two tables suggest that we can approximate the time variability of $r_t = [r(t, t+3m), \ldots, r(t, t+10y)]'$ in the following way:

$$r_t \approx \widehat{\mu} + \widehat{\gamma}_1 \mathbf{y_{1,t}} + \widehat{\gamma}_2 \mathbf{y_{2,t}} + \widehat{\gamma}_3 \mathbf{y_{3,t}}$$

- Same conclusions are obtained for yield variations (Δr_t) [see Piazzesi (2003)]. If we plot these 3 eigenvectors and the time series of the three *PCs* we find :

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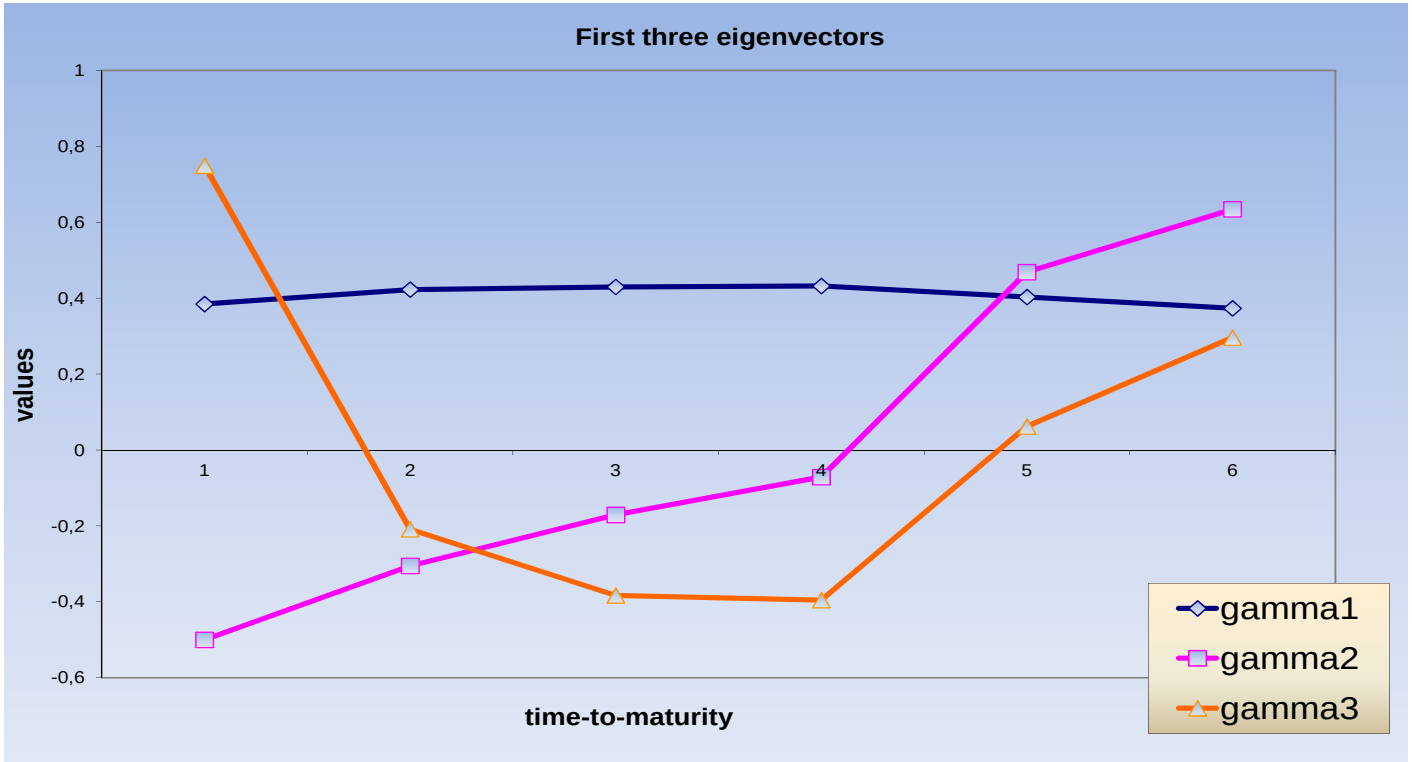

Figure 9: The first 3 eigenvectors of GSW data set.

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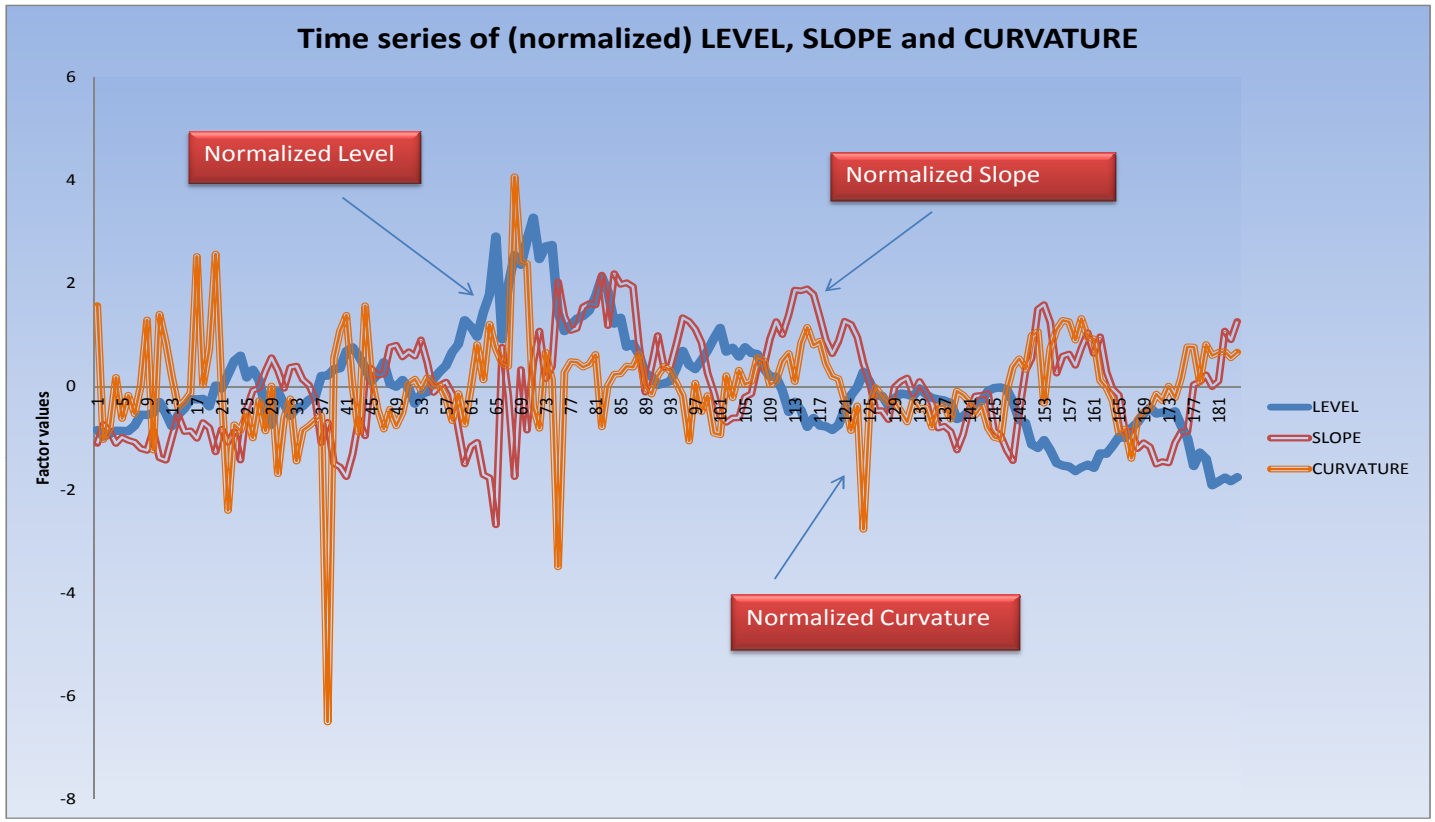

Figure 10: The first 3 PCs of GSW data set.

I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Nelson and Siegel (1987)

- Exponential-Polynomial families used by most central banks to construct the term structure of interest rates (see Lecture 2).

- The main feature is to capture (to fit) as much as possible the structure of market interest rates, at any point in time, with a small number of parameters. The Nelson and Siegel (1987) yield curve is parameterized as :

$$\begin{aligned} r(t, t+h) &= \beta_0 + \beta_1 \frac{1 - e^{(-h/\theta)}}{h/\theta} + \beta_2 \frac{1 - \left(1 + \frac{h}{\theta}\right) e^{(-h/\theta)}}{h/\theta} \\ &= \beta_0 \times B_0(h) + \beta_1 \times B_1(h) + \beta_2 \times B_2(h) \end{aligned}$$

where $B_0(h) = 1$,

$$B_1(h) = \frac{1 - e^{(-h/\theta)}}{h/\theta},$$

$$B_2(h) = \frac{1 - \left(1 + \frac{h}{\theta}\right) e^{(-h/\theta)}}{h/\theta}.$$

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Nelson and Siegel (1987)

- Interpretation of the parameters (estimated at each date t in the sample):

$$\begin{aligned}
\theta &= \text{scale (or "hump" location) parameter}\,, \\
\beta_0 &= \text{level parameter, i.e. } \beta_{0,t} \text{ is the level factor}\,, \\
\beta_1 &= \text{slope parameter, i.e. } \beta_{1,t} \text{ is the negative of the slope factor}\,, \\
\beta_2 &= \text{curvature parameter, i.e. } \beta_{2,t} \text{ is the curvature factor}\,.
\end{aligned}$$

- Indeed

$$\begin{aligned}
&(i) \quad \text{if } h \to +\infty \text{ (and } \theta > 0) \text{ then } r(t, t+h) \to \beta_0\,, \text{ (long-term yield)} \\
&(ii) \quad \text{if } h \to 0 \text{ then } r(t, t+h) \to (\beta_0 + \beta_1) = \text{(very) short rate} \\
&\Rightarrow \quad \beta_1 \text{ it drives the yield curve spread between}\,.
\end{aligned}$$

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Nelson and Siegel (1987)

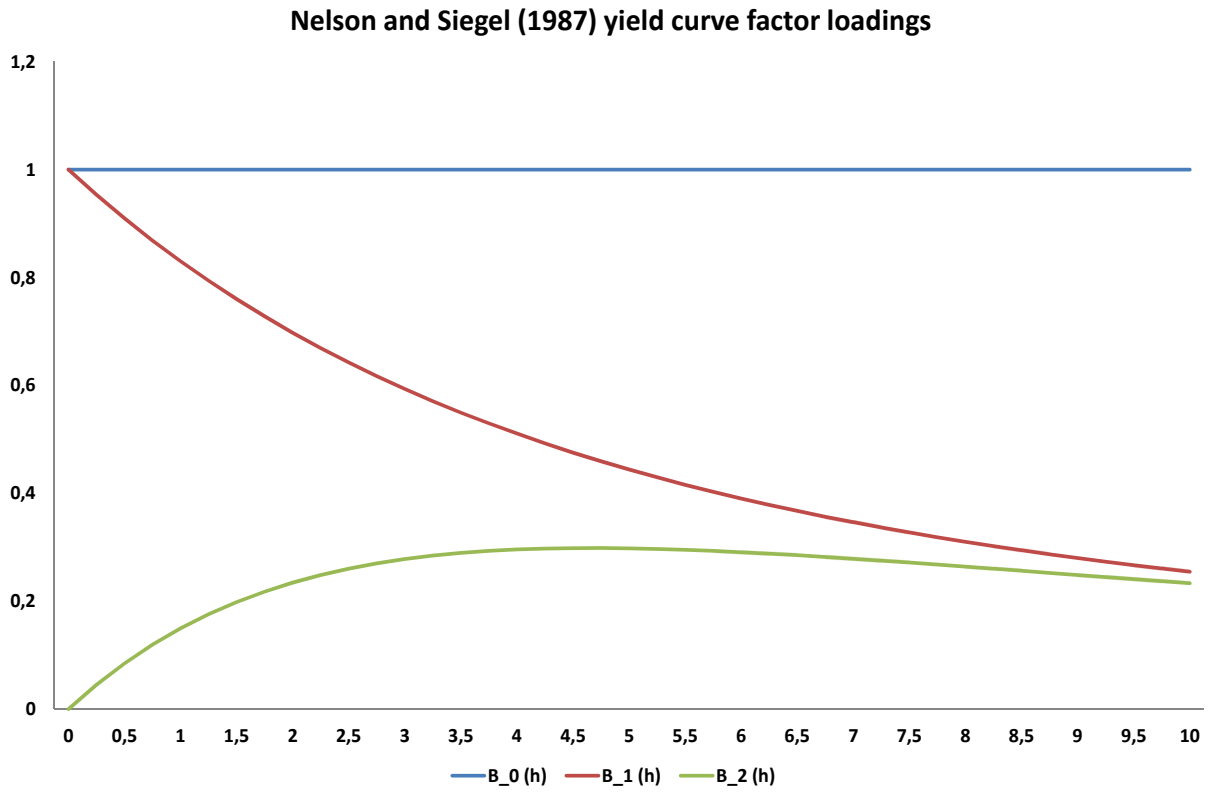

Figure 11: Nelson and Siegel (1987) yield curve loadings ($\theta = 2.6$).

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Svensson (1994)

- Svensson (1994) generalizes the Nelson-Siegel curve adding a third term.
- Precisely, he assumes:

$$r(t, t+h) = \beta_0 + \beta_1 \frac{1 - e^{(-h/\theta_1)}}{h/\theta_1} + \beta_2 \frac{1 - \left(1 + \frac{h}{\theta_1}\right) e^{(-h/\theta_1)}}{h/\theta_1} + \beta_3 \frac{1 - \left(1 + \frac{h}{\theta_2}\right) e^{(-h/\theta_2)}}{h/\theta_2} .$$

- The new parameters β_3 and θ_2 provide additional flexibility, compared with the Nelson-Siegel curve, and in particular over medium- to long-term maturities.
- In particular, θ_2 allows for a second "hump" in the yield curve.

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Svensson (1994)

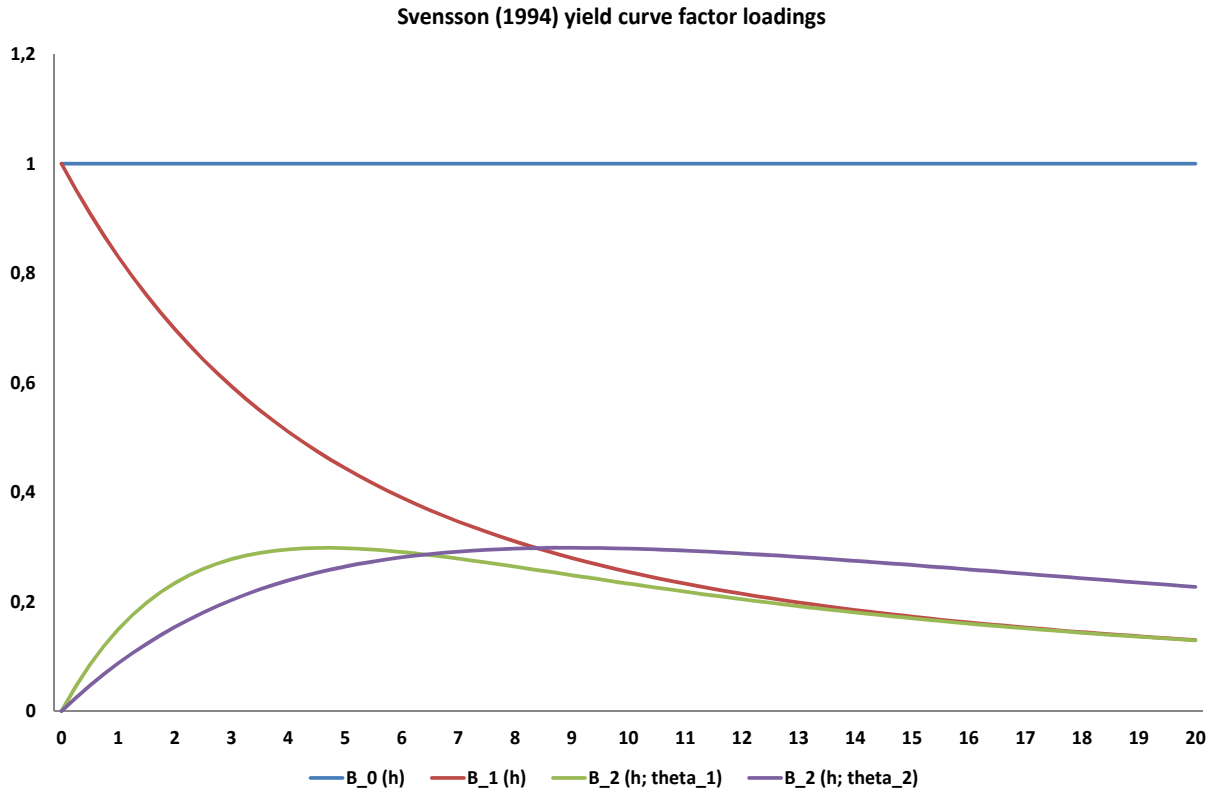

Figure 12: Svensson (1994) yield curve loadings.

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Svensson (1994)

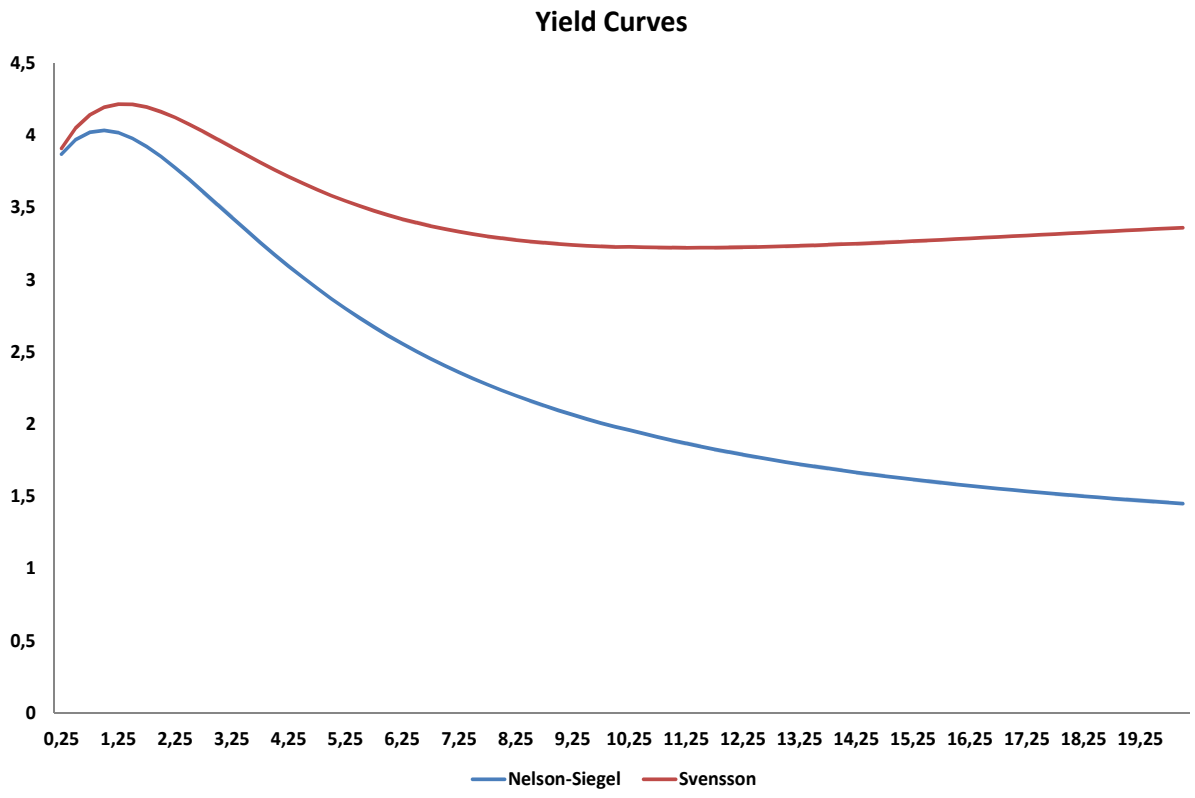

Figure 13: Nelson-Siegel and Svensson yield curves.

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Diedold and Li (2006)

- Purpose : forecasting the term structure of Government bond yields. DL (2006) assume (different notation):

$$r(t, t+h) \quad = \quad \beta_{1,t} + \beta_{2,t} \frac{1 - e^{(-h\,/\,\theta_t)}}{h\,/\,\theta_t} + \beta_{3,t} \frac{1 - \left(1 + \frac{h}{\theta_t}\right) e^{(-h\,/\,\theta_t)}}{h\,/\,\theta_t} \,.$$

- As we have previously seen, parameters could be estimated at each date by NLLSs. Instead, here DL (2006) fix θ_t to an *appropriate value* and then estimate $(\beta_{1,t}, \beta_{2,t}, \beta_{3,t})$ by OLS using as observations date-t (ZCB) yields.

- What an appropriate value of θ_t is ? θ_t determines the maturity at which the loading on the curvature factor achieves its maximum. Two- or three-year maturities are commonly used in that regard, so they take the average 30 months.

- The θ_t value that maximizes the loading on the curvature factor at exactly 30 months is $\theta_t = 0.0609$.

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Diedold and Li (2006)

F.X. Diebold, C. Li / Journal of Econometrics 130 (2006) 337–364 347

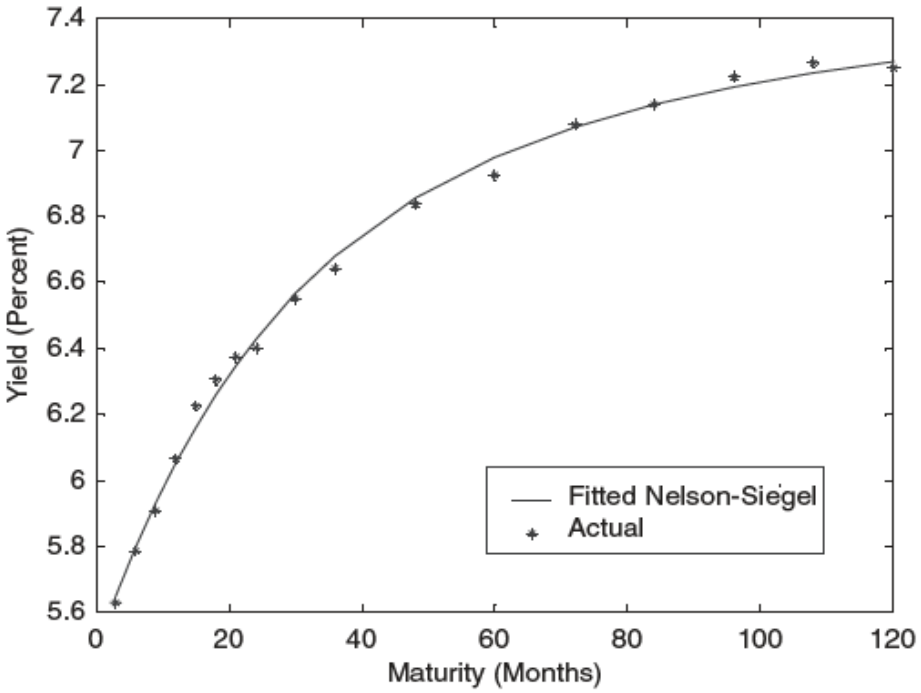

Fig. 4. Actual (data-based) and fitted (model-based) average yield curve. We show the actual average yield curve and the fitted average yield curve obtained by evaluating the Nelson–Siegel function at the mean values of $\hat{\beta}_{1t}$, $\hat{\beta}_{2t}$ and $\hat{\beta}_{3t}$ from Table 3.

Figure 14: Diedold and Li (2006).

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Diedold and Li (2006)

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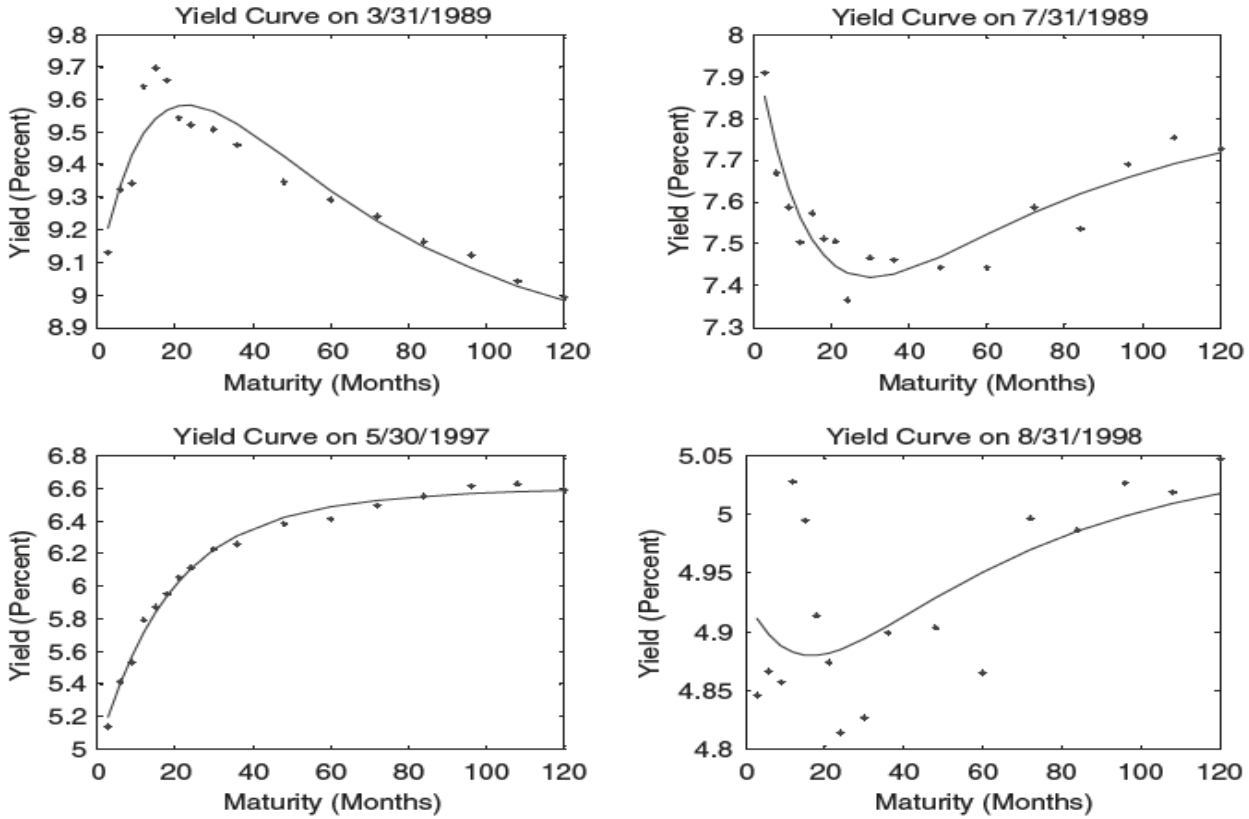

Fig. 5. Selected fitted (model-based) yield curves. We plot fitted yield curves for selected dates, together with actual yields. See text for details.

Figure 15: Diedold and Li (2006).

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Diedold and Li (2006)

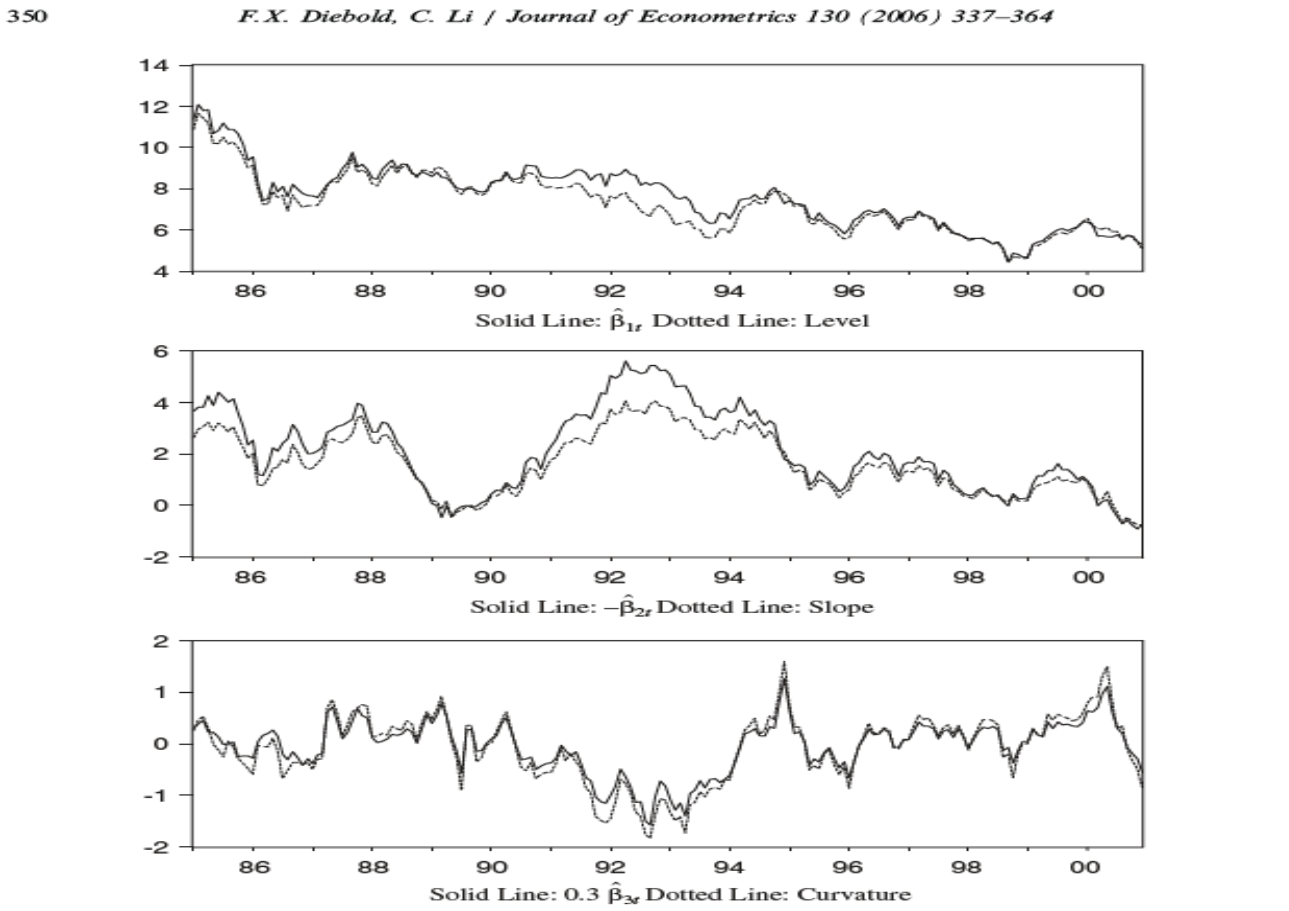

Fig. 7. Model-based level, slope and curvature (i.e., estimated factors) vs. data-based level, slope and curvature. We define the level as the 10-year yield, the slope as the difference between the 10-year and 3-month yields, and the curvature as the twice the 2-year yield minus the sum of the 3-month and 10-year yields.

Figure 16: Diedold and Li (2006).

I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Diedold, Rudebusch and Aruoba (2006)

- DRA (2006) assume:

$$r(t, t+h) \quad = \quad L_t + S_t \, \frac{1 - e^{(-h\,/\,\theta)}}{h\,/\,\theta} + C_t \, \frac{1 - \left(1 + \frac{h}{\theta}\right) e^{(-h\,/\,\theta)}}{h\,/\,\theta} \, ,$$

where L_t, S_t and C_t are the latent level, slope and curvature factors.

- The joint dynamics of $w_t = (L_t, S_t, C_t)'$ is described by the following Gaussian VAR(1) process (transition equation):

$$(w_t - \mu_w) = A \, (w_{t-1} - \mu_w) + \eta_{w,t} \, ,$$

with $\eta_{w,t} \sim IIN(0, Q)$. Given the observed yields $R_t = (r(t, t+h_1), \ldots, r(t, t+h_n))'$, the measurement equations linking the yields to the latent factors are given by:

$$R_t = \begin{bmatrix} 1 & \frac{1-e^{(-h_1\,/\,\theta)}}{h_1\,/\,\theta} & \frac{1-\left(1+\frac{h_1}{\theta}\right)e^{(-h_1\,/\,\theta)}}{h_1\,/\,\theta} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{(-h_n\,/\,\theta)}}{h_n\,/\,\theta} & \frac{1-\left(1+\frac{h_n}{\theta}\right)e^{(-h_n\,/\,\theta)}}{h_n\,/\,\theta} \end{bmatrix} \begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{n,t} \end{bmatrix} \, ,$$

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I.10 – EXPONENTIAL-POLYNOMIAL FAMILIES

Diedold, Rudebusch and Aruoba (2006)

- The associated state-space system can be written (with obvious notation) as follows:

$$
\begin{aligned}
(w_t - \mu_w) &= A\,(w_{t-1} - \mu_w) + \eta_{w,t}\,, \\
R_t &= \Lambda\, w_t + \varepsilon_{R,t}\,,
\end{aligned}
$$

where

$$
\begin{pmatrix} \eta_{w,t} \\ \varepsilon_{R,t} \end{pmatrix} \sim IIN\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right].
$$

- This is a linear Gaussian State-Space model, and it is estimated by Maximum Likelihood, with a likelihood function calculated by Kalman Filter recursions able to properly take into account the latent nature of transition factors.

- An extension of this yields-only model is obtained by introducing in the vector of state variables w_t observable macro-financial variables :

- the level of real economic activity relative to potential (capacity utilization rate in manufacturing industry), the YoY inflation rate (from price deflator in the personal consumption expenditure), and the Federal Fund Rate.

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Chapter II

A NEW PERSPECTIVE ON GATSMs
- PART I -

– Agenda –

II.1 Introduction. (Slide 122)

II.2 A Gaussian model. (Slide 126)

II.3 Information in the cross-section. (Slide 133)

II.4 The role of invertibility in empirical analysis. (Slide 134)

II.5 A two-factor example. (Slide 136)

II.6 A canonical hidden-factor model. (Slide 139)

II.1 – INTRODUCTION

- **Gaussian ATSMs typically feature the following ingredients**:

(*i*) a vector of state variables (risk factors) following a Gaussian VAR(1) process (under $\mathbb{P}$ and $\mathbb{Q}$);

(*ii*) a low-dimensional factor-structure in which the risk factors are latent variables, yield-based variables (and, possibly, macroeconomic variables);

(*iii*) the assumption of no-arbitrage opportunities in bond markets.

(*iv*) a risk-free short rate affine function of the state variables.

- **As far as the second condition is concerned**:

  - the low dimension is motivated by the small (two or three) number of principal components (PCs) required to explain the variation of bond yields. Macro variables (if any) are typically economic activity and inflation rate.

- **Under the previous standard conditions**:

  - theoretical yields are affine functions of *all* state variables;
  - the state vector driving the TS can be represented as a function of YIELDS, by inversion of the affine YTM formula.

II.1 – INTRODUCTION

- The purpose of this Chapter (based on Duffee (2011)) is to show that this assumption rules out an important class of term structure dynamics.

- This standard approach implies that today's TS contains all of the information relevant to predicting both future returns to bonds and future bond yields. The associated empirical literature has exploited this Markov structure in many ways:

a) To choose the dimension of the model, given that the same factors that determine the cross-section of yields also determine yields dynamics.

b) Time-t conditional expectations of returns can be expressed entirely in terms of forward rates observed at the same date. Other data are unnecessary to model yield dynamics.

c) The one-to-one mapping from factors to yields leads to tractable estimation procedure.

II.1 – INTRODUCTION

- Recent empirical literature calls this assumption into question.
  - Priestly (2009) and Ludvigson and Ng (2009) find that various measure of macro activity contain information about excess bond returns that is not in current forward rates.
  - Cochrane and Piazzesi (2005) find that lagged forward rates contain information about future excess bond returns that is not in current forward rates.

- We will see that it easy to build a multifactor (Gaussian affine) term structure model in which:
  - One of the factors plays an important role in determining investors' expectations of future yields (expectation component), but at the same time it has no effects on current yields. It is named "hidden factor".
  - This means that a hidden factor has opposite effects on expected future short rates and bond risk (term) premia in such a way that the associated yields' variations is negligible.
  - More generally, factors that drive risk premia and expected short rates in opposite directions can have arbitrarily small effects on the cross-section of yields, yet large effects on yield dynamics.

II.1 – INTRODUCTION

- This theoretical results can be inferred from existing term structure literature :

  - Duffee (2002) gives an example in which the physical and the risk-neutral dynamics are driven by state vectors with different dimensions.
  - Its implications for the empirical work have not been recognized until Duffee (2011) and the independent work of Joslin, Priebsch and Singleton (2014, JF).
  - Duffee (2011) considers latent factors and use filtering to examine whether there are hidden factor in the observed US yield curve.
  - JPS (2014), following regression-based empirical evidence, assume the existence of two hidden factors that are linear combinations of observed inflation and economic activity, and estimate the resulting model using both yields and macro variables.

II.2 – A GAUSSIAN ATSM

- Let us consider a Gaussian ATSM where the K-dimensional state vector w_t is made of latent variables (identification conditions are thus required) and follows under the $\mathbb{P}$-measure the following Gaussian VAR(1) process:

$$w_{t+1} = \nu + \Phi\, w_t + \Sigma\, \varepsilon_{t+1}\,, \tag{II.1}$$

where $\varepsilon_{t+1} \,|\, \underline{w_t} \sim N(0, I_K)$. The continuously compounded risk-free short rate r_t is such that:

$$r_t = \delta_0 + \delta_1' w_t\,. \tag{II.2}$$

- The log-SDF is given by:

$$\log M_{t,t+1} = -r_t - \Lambda_t\, \varepsilon_{t+1} - \frac{1}{2}\Lambda_t'\Lambda_t\,. \tag{II.3}$$

- The vector Λ_t is the compensation investors require to face shocks to the state vector and it is given by (see Duffee (2002)):

$$\Sigma\, \Lambda_t = \lambda_0 + \lambda_1\, w_t \tag{II.4}$$

II.2 – A GAUSSIAN ATSM

- Under the (equivalent martingale) $\mathbb{Q}$-measure the state vector is Gaussian VAR(1):

$$w_{t+1} = \nu^q + \Phi^q \, w_t + \Sigma \, \varepsilon^q_{t+1} \,, \tag{II.5}$$

where risk-neutral parameters are:

$$\nu^q = \nu - \lambda_0 \,, \quad \Phi^q = \Phi - \lambda_1 \,. \tag{II.6}$$

- The affine yield-to-maturity formula is given by :

$$R(t,h) = -\frac{1}{h} \left(C_h' w_t + D_h \right) = \mathcal{C}_h' w_t + \mathcal{D}_h \,, \tag{II.7}$$

where

$$\begin{cases} C_h = -\delta_1 + (\Phi^q)' \, C_{h-1} \,, \quad C_0 = 0 \,, \\ D_h = -\delta_0 + C_h' \nu^q + \frac{1}{2} \, C_{h-1}' (\Sigma \, \Sigma') \, C_{h-1} + D_{h-1} \,, \quad D_0 = 0 \,. \end{cases} \tag{II.8}$$

- The excess log return to a h-period bond over j periods is defined as:

$$xr^{(h)}_{t,t+j} = -(h-j) \, R(t+j, h-j) + h \, R(t,h) - j \, R(t,j) \,. \tag{II.9}$$

II.2 – A GAUSSIAN ATSM

- $xr^{(h)}_{t,t+j}$ can be written as:

$$
\begin{aligned}
xr^{(h)}_{t,t+j} &= [h\,\mathcal{D}_h - (h-j)\,\mathcal{D}_{h-j} - j\,\mathcal{D}_j] + [h\,\mathcal{C}'_h - j\,\mathcal{C}'_j]\,w_t \\
&\quad -(h-j)\,\mathcal{C}'_{h-j} w_{t+j} \\
&= [h\,\mathcal{D}_h - (h-j)\,\mathcal{D}_{h-j} - j\,\mathcal{D}_j] + [h\,\mathcal{C}'_h - j\,\mathcal{C}'_j]\,w_t \\
&\quad -(h-j)\,\mathcal{C}'_{h-j}\left[(I+\Phi+\ldots+\Phi^{j-1})\,\nu + \Phi^j\,w_t + \Sigma\sum_{i=0}^{j-1}\Phi^i\,\varepsilon_{t+j-1}\right] \\
&= [h\,\mathcal{D}_h - (h-j)\,\mathcal{D}_{h-j} - j\,\mathcal{D}_j] + [h\,\mathcal{C}'_h - j\,\mathcal{C}'_j - (h-j)\,\mathcal{C}'_{h-j}\Phi^j]\,w_t \\
&\quad -(h-j)\,\mathcal{C}'_{h-j}\left[(I+\Phi+\ldots+\Phi^{j-1})\,\nu\right] - (h-j)\,\mathcal{C}'_{h-j}\Sigma\sum_{i=0}^{j-1}\Phi^i\,\varepsilon_{t+j-1} \\
&\quad \pm[h\,\mathcal{C}'_h - j\,\mathcal{C}'_j]\,[I-\Phi]^{-1}\,\nu\,.
\end{aligned}
\tag{II.10}
$$

- The 3rd term on the RHS can be written as:

$$
\begin{aligned}
&-(h-j)\,\mathcal{C}'_{h-j}\left[\sum_{i=0}^{j-1}\Phi^j\right]\nu = -(h-j)\,\mathcal{C}'_{h-j}(I-\Phi^j)\,(I-\Phi)^{-1}\,\nu \\
= \quad & -(h-j)\,\mathcal{C}'_{h-j}\mathbb{E}(w_t) + (h-j)\,\mathcal{C}'_{h-j}\Phi^j\,\mathbb{E}(w_t)\,.
\end{aligned}
$$

II.2 – A GAUSSIAN ATSM

- $xr_{t,t+j}^{(h)}$ can therefore be written as:

$$
\begin{aligned}
xr_{t,t+j}^{(h)} \quad = \quad & \left[h\,\mathcal{D}_h - (h-j)\,\mathcal{D}_{h-j} - j\,\mathcal{D}_j + \left(h\,\mathcal{C}_h' - j\,\mathcal{C}_j' - (h-j)\,\mathcal{C}_{h-j}' \right) \mathbb{E}(w_t) \right] \\
& + \left[h\,\mathcal{C}_h' - j\,\mathcal{C}_j' - (h-j)\,\mathcal{C}_{h-j}' \Phi^j \right] (w_t - \mathbb{E}(w_t)) \\
& -(h-j)\,\mathcal{C}_{h-j}' \Sigma \left[\sum_{i=0}^{j-1} \Phi^i\, \varepsilon_{t+j-1} \right] .
\end{aligned}
\tag{II.11}
$$

- The first term in square brackets on the RHS is the unconditional mean of $xr_{t,t+j}^{(h)}$.
- The second term is the conditional deviation from the unconditional mean.
- The third term is the return shock.

II.2 – A GAUSSIAN ATSM

- Given the latent nature of the state vector w_t, our GATSM specified by relations (II.1), (II.2) and (II.5) can be translated and rotated without observable consequences.
- More precisely, as shown by Dai and Singleton (2000), the application of the invariant transformation $\tilde{w}_t = A + B\, w_t$ gives an observationally equivalent term structure model (i.e., identical short rate and bond price processes).

- Indeed, with respect to the state $\tilde{w}_t$, and from $w_t = B^{-1}(\tilde{w}_t - A)$, the $\mathbb{P}$-dynamics is Gaussian VAR(1):

$$\begin{aligned} B^{-1}(\tilde{w}_{t+1} - A) &= \nu + \Phi\, B^{-1}(\tilde{w}_t - A) + \Sigma\, \varepsilon_{t+1} \\ \tilde{w}_{t+1} &= \left[\left(I - (B\,\Phi\,B^{-1})\right) A + B\,\nu\right] + (B\,\Phi\,B^{-1})\,\tilde{w}_t + B\,\Sigma\,\varepsilon_{t+1} \\ &= \tilde{\nu} + \tilde{\Phi}\,\tilde{w}_t + \tilde{\Sigma}\,\varepsilon_{t+1}\,. \end{aligned} \tag{II.12}$$

- Similarly, under the $\mathbb{Q}$ measure, $\tilde{w}_t$ is Gaussian VAR(1) with: $\tilde{\nu}^{\mathbb{Q}} = \left(I - (B\,\Phi^{\mathbb{Q}}\,B^{-1})\right) A + B\,\nu^{\mathbb{Q}}$, $\tilde{\Phi}^{\mathbb{Q}} = B\,\Phi^{\mathbb{Q}}\,B^{-1}$, $\tilde{\Sigma} = B\,\Sigma$.
- We have $r_t = \tilde{\delta}_0 + \tilde{\delta}_1'\,\tilde{w}_t$ with $\tilde{\delta}_0 = \delta_0 - \delta_1'\,B^{-1}\,A$ and $\tilde{\delta}_1 = \left(B^{-1}\right)'\,\delta_1$.

II.2 – A GAUSSIAN ATSM

- Regarding the observational equivalence provided by the invariant transformation $\tilde{w}_t = A + B\, w_t$, we have:

- $\tilde{\delta}_0 + \tilde{\delta}_1' \tilde{w}_t = \delta_0 + \delta_1' w_t = r_t$ and (with obvious notation)

- $C_h(\delta_1, \Phi^{\mathbb{Q}})' w_t + D_h(\delta_0, \nu^{\mathbb{Q}}, \Sigma\Sigma') = \tilde{C}_h(\tilde{\delta}_1, \tilde{\Phi}^{\mathbb{Q}})' \tilde{w}_t + \tilde{D}_h(\tilde{\delta}_0, \tilde{\nu}^{\mathbb{Q}}, \tilde{\Sigma}\tilde{\Sigma}')$,

  with $C_h = B' \tilde{C}_h$ and $D_h = \tilde{D}_h + (\tilde{C}_h)' A$.

- Thus, the GATSM with state vector $\tilde{w}_t$ and parameters $(\tilde{\nu}^{\mathbb{Q}}, \tilde{\Phi}^{\mathbb{Q}}, \tilde{\Sigma}, \tilde{\delta}_0, \tilde{\delta}_1)$ is observationally equivalent

- to the GATSM with state vector w_t and parameters $(\nu^{\mathbb{Q}}, \Phi^{\mathbb{Q}}, \Sigma, \delta_0, \delta_1)$, since they provide identical (observable) short-rate and bond price processes.

II.2 – A GAUSSIAN ATSM

- A canonical Gaussian affine yield curve model is a GATSM that is maximally flexible in its parameterization of the $\mathbb{P}$- and $\mathbb{Q}$-dynamics of the latent state vector w_t, but subject to no-arbitrage conditions and to normalization restrictions to guarantee econometric identification.

- For example, Dai and Singleton (2000) identify the canonical representation of GATSMs by imposing $\nu = 0$, Φ a lower triangular matrix and $\Sigma = \text{Id}$, and $\delta_1 \geq 0$.

- Hamilton and Wu (2012) consider similar conditions, but $\Phi^{\mathbb{Q}}$ lower triangular instead of Φ, while Singleton (2006) takes $\Sigma = \text{Id}$, $\delta_1 \geq 0$, $\nu^{\mathbb{Q}} = 0$ and $\Phi^{\mathbb{Q}}$ lower triangular.

- Those $K(K+1)$ restrictions fix the free parameters that characterize the invariant transformation $A + Bw_t$. Observe also that the latent nature of the state variables precludes imposing any ICC.

- [Next Lecture: the case of a state vector made of (observable) linear combination of yields and macro variables.]

II.3 – INFORMATION IN THE CROSS-SECTION

- Absent specific parameter restrictions, the date-t state vector w_t can be inferred from a cross-section of date-t yields. More formally, if we collect the yields on $K = \dim(w_t)$ ZCBs in the vector $R_t^{(cs)}$, then this vector can be written:

$$R_t^{cs} = \mathcal{C}^{cs} \cdot w_t + \mathcal{D}^{cs} \,, \tag{II.13}$$

where $\dim(\mathcal{D}^{cs}) = K$, with $\mathcal{D}^{cs} = (\mathcal{D}_{h_1}, \ldots, \mathcal{D}_{h_K})'$, while $\mathcal{C}^{cs}$ is a $(K \times K)$ matrix with rows $\mathcal{C}'_{h_i}$ for each relevant bond maturity h_i.

- Assuming $(\mathcal{C}^{cs})^{-1}$ exists, the selected vector of yields contains the same information as w_t:

$$w_t = (\mathcal{C}^{cs})^{-1} \left(R_t^{cs} - \mathcal{D}^{cs}\right), \tag{II.14}$$

and given that w_t follows a Gaussian VAR(1) process, the same is for R_t^{cs}. Indeed, we have:

$$\begin{aligned}
(\mathcal{C}^{cs})^{-1} \left(R_{t+1}^{cs} - \mathcal{D}^{cs}\right) &= \nu + \Phi \left[(\mathcal{C}^{cs})^{-1} \left(R_t^{cs} - \mathcal{D}^{cs}\right)\right] + \Sigma\, \varepsilon_{t+1} \\
R_{t+1}^{cs} &= \left[\mathcal{C}^{cs}\, \nu + \mathcal{D}^{cs} - \mathcal{C}^{cs}\, \Phi\, (\mathcal{C}^{cs})^{-1} \mathcal{D}^{cs}\right] + \left[\mathcal{C}^{cs}\, \Phi\, (\mathcal{C}^{cs})^{-1}\right] R_t^{cs} + \mathcal{C}^{cs}\, \Sigma\, \varepsilon_{t+1} \\
&= \nu^{cs} + \Phi^{cs}\, R_t^{cs} + \Sigma^{cs}\, \varepsilon_{t+1} \,.
\end{aligned} \tag{II.15}$$

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II.4 – THE ROLE OF INVERTIBILITY

- The invertibility of $\mathcal{C}^{cs}$ allows the inference of $K = \dim(w_t)$ from the properties of the cross-section of yields.

- Factor analysis of the unconditional variance-covariance matrix of yields (or first-difference of yields) allows to infer K.
  - Litterman and Scheinkman (1991) conclude that $K = 3$ factors all but a negligible fraction of the variation of the term structure.

- Relation (II.15) implies that maximum likelihood estimation of GATSMs requires only a panel of K yields and the density function of the state vector.
  - Pearson and Sun (1994) are the first to exploit this result.
  - Chen and Scott (1994) expand this result assuming that K linear combination of yields are observed without errors, while $L - K$ are observed with error.

II.4 – THE ROLE OF INVERTIBILITY

- In the case of a Gaussian model, MLE is also feasible when all L yields are observed with measurement error (via Kalman filter).
- Invertibility implies that only yields are necessary to estimate our model, but it does not rule out the use of other (macro) data.
- Although invertibility is widely assumed and useful, it need not hold.
- Let us now consider special cases of the Gaussian affine framework where $\text{rank}(\mathcal{C}^{cs}) < K$, so that the state vector cannot be extracted from the cross-section of yields. A two-factor example illustrates the mathematics and the economic intuition behind.

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II.5 – A TWO-FACTOR EXAMPLE

- Let us consider a 2-factor Gaussian affine model where the state vector is given by $w_t = (r_t, h_t)'$.
- The first component (r_t) is the risk-free short rate, while the second one (h_t) is a latent variable that will play the role of a *hidden factor*.
- The historical dynamics is given by:

$$\begin{bmatrix} r_{t+1} \\ h_{t+1} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} + \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} r_t \\ h_t \end{bmatrix} + \Sigma \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}. \tag{II.16}$$

- Observe that, when $\varphi_{12} \neq 0$, $\mathbb{E}[r_{t+k} \mid r_t, h_t]$ depend on *both* r_t and h_t. This means that h_t captures all available information about expected future short rates not captured by r_t.
- If both $R^{EH}(t,h)$ and $TP_{t,h}$ components of any given yield depend on h_t (and it is in general the case), the net effect of h_t on yields in ambiguous.
- The restriction we adopt here in this example is that changes in $TP_{t,h}$ exactly cancel changes in $R^{EH}(t,h)$, leaving yields unaffected by h_t.

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II.5 – A TWO-FACTOR EXAMPLE

- Formally, the requirement is $\varphi_{12}^{\mathbb{Q}} = 0$ or, equivalently, $\varphi_{12} = \lambda_{1,12}$. This means that the risk-neutral dynamics of the state variable is:

$$\begin{bmatrix} r_{t+1} \\ h_{t+1} \end{bmatrix} = \begin{bmatrix} \nu_1^{\mathbb{Q}} \\ \nu_2^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} \varphi_{11}^{\mathbb{Q}} & 0 \\ \varphi_{21}^{\mathbb{Q}} & \varphi_{22}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} r_t \\ h_t \end{bmatrix} + \Sigma \begin{bmatrix} \varepsilon_{1,t}^{\mathbb{Q}} \\ \varepsilon_{2,t}^{\mathbb{Q}} \end{bmatrix}. \tag{II.17}$$

We observe that r_t is an autonomous (Markovian) Gaussian AR(1) process under $\mathbb{Q}$. In the YTM formula $R(t,h) = \mathcal{C}_h' w_t + \mathcal{D}_h$, we have:

$$\mathcal{C}_h = (\mathcal{C}_{1,h}, 0)',$$

with $\mathcal{C}_{1,h} = \frac{1}{h}(1 - \varphi_{11}^{\mathbb{Q}})^{-1}\left(1 - (\varphi_{11}^{\mathbb{Q}})^h\right)$.

- In this case the matrix $\mathcal{C}^{cs}$ cannot be inverted (contrary to the previous general framework).
- The factor h_t is *hidden* in the sense that it does not enter the YTM formula anymore, and therefore it cannot be inferred from the cross-section of the date-t yields.

II.5 – A TWO-FACTOR EXAMPLE

- Nevertheless, investors observe h_t and they take it into account when setting bond prices and forming expectations of future yields or future returns on bonds.
- Indeed, if we consider the one-period expected excess return on a bond with maturity h at t we have:

$$\begin{aligned}
\mathbb{E}_t[xr^{(h)}_{t,t+1}] &= h\,R(t,h) - (h-1)\,\mathbb{E}_t[R(t+1,h-1)] - r_t \\
&= h\,\mathcal{D}_h - (h-1)\,\mathcal{D}_{h-1} \\
&\quad +(1-\varphi^{\mathbb{Q}}_{11})^{-1}\left[\left(1-(\varphi^{\mathbb{Q}}_{11})^h\right) - \left(1-\varphi^{\mathbb{Q}}_{11})^{h-1}\right)\varphi_{11} - 1\right] r_t \\
&\quad -(1-\varphi^{\mathbb{Q}}_{11})^{-1}\left(1-(\varphi^{\mathbb{Q}}_{11})^{h-1}\right)\varphi_{12}\,h_t\,.
\end{aligned} \tag{II.18}$$

- The final term captures the dependence of expected excess returns on h_t. We observe that, with $w_{1,t} = r_t$, a single-parameter restriction is required to generate this structure.
- Armed with the intuition of this example, it is straightforward to proceed to the more general case in which the short rate follows a K-factor Markov process under the $\mathbb{P}$ measure and an K_0-factor Markov process under the equivalent martingale measure, where $K_0 < K$.

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II.6 – A CANONICAL HIDDEN-FACTOR MODEL

- The identification problem, associated to the invariant transformation presented in II.2, is solved by imposing $K(K+1)$ parameter restrictions required by the degrees of freedom introduced by A and B.

- For instance, Joslin, Singleton and Zhu (2011) consider $\nu^{\mathbb{Q}} = 0$, $\Phi^{\mathbb{Q}} = \text{diag}(\lambda^{\mathbb{Q}})$ (a diagonal matrix with real distinct eigenvalues) and $\delta_1 = \mathbf{1}$. Given that this latter condition prevents from having hidden factors, it will be replaced by another one.

- With this kind of rotation, we have $w_{i,t} = h_{i,t}$ if $\delta_{1,i} = 0$ (being $\Phi^{\mathbb{Q}}$ diagonal). Let us assume to have $\delta_1 = (\delta'_{1,f}, \delta'_{1,h})'$, with:

$$\begin{aligned} \dim(\delta_{1,f}) &= K_f\,, \quad \text{and} \quad \delta_{1,f} \neq 0\,, \\ \dim(\delta_{1,h}) &= K_h\,, \quad \text{and} \quad \delta_{1,h} = 0\,; \end{aligned}$$

  then we can write $w_t = (f'_t, h'_t)'$, with $\dim(f_t) = K_f$ and $\dim(h_t) = K_h$.

- The parameters of the $\mathbb{P}$-dynamics are $\nu = \lambda_0$ and $\Phi = \text{diag}(\lambda^{\mathbb{Q}}) + \lambda_1$.

II.6 – A CANONICAL HIDDEN-FACTOR MODEL

- Fixed-income investors require compensation to face shocks to f_t ($\lambda_{t,f} \neq 0$). This required compensation will vary with h_t if the necessary elements of λ_1 are nonzero. This channel also allows h_t to contain information about the evolution of the short rate that is not in f_t.
- No fixed-income instrument has a price innovation driven by innovations in h_t. Thus, although investors may require compensation to face shocks to h_t, no information about this compensation is available in the fixed-income market.
- Hence, a canonical model of fixed income can normalize risk premia on h_t to zero, which is why the dynamics of h_t under the equivalent martingale measure equal those under the physical measure.

- Final normalizations for a canonical model are scaling normalizations:

a) The non-hidden vector f_t is scaled by normalizing $\delta_{1,f} = \mathbf{1}$.

b) The hidden vector h_t features $\delta_{1,h} = \mathbf{0}$.

II.6 – A CANONICAL HIDDEN-FACTOR MODEL

- The author estimate a five-factor Gaussian affine term structure model using monthly Treasury yields from 1964 to 2007.
  - The model's risk premia dynamics are parsimonious: a single "risk premium factor" determines the one-month-ahead risk premia on all bonds.
  - Half of the variation in expected excess bond returns is explained by a hidden component.
  - The case for the importance of hidden components of risk premia is strong both statistically and economically.

- Measures of macroeconomic activity explain only a small fraction of variation in the hidden factor.
  - This result sharply distinguishes this estimated model from the structure imposed by JPS (2014).
  - Similarly, the ability of the hidden factor to forecast excess returns is not captured, in a regression sense, by other macroeconomic or financial variables.